

# ON THE HEEGAARD FLOER HOMOLOGY OF BRANCHED DOUBLE-COVERS

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ABSTRACT. Let  $L \subset S^3$  be a link. We study the Heegaard Floer homology of the branched double-cover  $\Sigma(L)$  of  $S^3$ , branched along  $L$ . When  $L$  is an alternating link,  $\widehat{HF}$  of its branched double-cover has a particularly simple form, determined entirely by the determinant of the link. For the general case, we derive a spectral sequence whose  $E^2$  term is a suitable variant of Khovanov's homology for the link  $L$ , converging to the Heegaard Floer homology of  $\Sigma(L)$ .

## 1. INTRODUCTION

Given a link  $L \subset S^3$ , we can form its branched double cover, a new three-manifold which we denote by  $\Sigma(L)$ . In this paper, we study the Heegaard Floer homology of this three-manifold  $\widehat{HF}(\Sigma(L))$  (c.f. [17]).

The starting point for these investigations is a skein exact sequence which this link invariant  $L \mapsto \widehat{HF}(\Sigma(L))$  satisfies. Specifically, fix a projection of  $L$ , and let  $L_0$  and  $L_1$  denote the two resolutions of  $L$  at a crossing for the projection, as illustrated in Figure 1. It is a quick consequence of the surgery long exact sequence for  $\widehat{HF}$  that for any link  $L \subset S^3$ , the groups  $\widehat{HF}(L)$ ,  $\widehat{HF}(L_0)$ , and  $\widehat{HF}(L_1)$  fit into a long exact sequence

$$(1) \quad \dots \longrightarrow \widehat{HF}(\Sigma(L_0)) \longrightarrow \widehat{HF}(\Sigma(L_1)) \longrightarrow \widehat{HF}(\Sigma(L)) \longrightarrow \dots$$

This skein exact sequence leads readily to a complete calculation of  $\widehat{HF}(\Sigma(L))$ , where  $L$  is any alternating link, explained in Section 3. In particular, it is shown there that if  $L$  is a link which admits a connected, alternating projection, then the rank of  $\widehat{HF}(\Sigma(L))$  agrees with the number of elements in  $H^2(\Sigma(L); \mathbb{Z})$ , i.e. that  $\Sigma(L)$  is what might be called an “ungraded Heegaard Floer homology lens space” or, in the terminology of [16], an  $L$ -space. (Note that this argument applies to a larger class of links which includes all alternating links as a special case.)

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*Key words and phrases.* Heegaard Floer homology, branched double-covers, Khovanov's homology, alternating links, four-ball genus.

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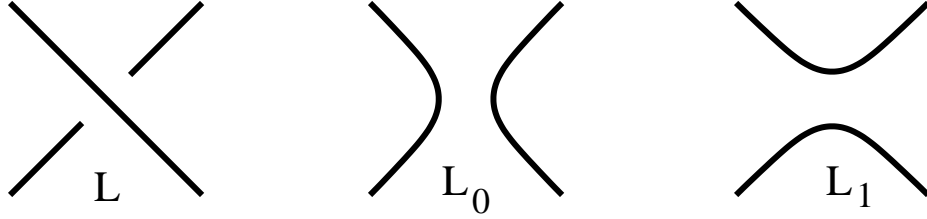


FIGURE 1. **Skein moves.** Given a link with a crossing as labelled in  $L$  above, we have two “resolutions”  $L_0$  and  $L_1$ , obtained by replacing the crossing by the two simplifications pictured above.

When  $Y$  is an arbitrary three-manifold,  $\widehat{HF}(Y)$  has the structure of a  $\mathbb{Z}/2\mathbb{Z}$ -graded Abelian group, and that is the structure we will be concerned with throughout most of this paper. But in general,  $\widehat{HF}(Y)$  also comes with a natural splitting into summands indexed by  $\text{Spin}^c$  structures on  $Y$  [17]. Indeed, when  $Y$  is a rational homology three-sphere, the groups are further endowed with an absolute  $\mathbb{Q}$ -grading [18].

By further elaborating on the calculations for  $\Sigma(L)$  when  $L$  is alternating, we are able to determine this extra structure explicitly from the alternating diagram for  $L$ , as explained in Section 3. As explained in [18] (compare also [7]), this structure gives constraints on the intersection forms of negative-definite four-manifolds which bound  $\Sigma(L)$ .

Turning back to the case of a general link  $L$ , it is suggestive to compare the Exact Sequence (1) with the work of Khovanov, c.f. [11] (for the reader’s convenience, we briefly review the construction in Section 5). Specifically, Khovanov introduces an invariant for links in  $S^3$  whose Euler characteristic, in a suitable sense, is the Jones polynomial (c.f. [9], see also [10]). By construction, his invariants satisfy a “skein exact sequence” inspired by the skein relation for the Jones polynomial. In particular, just like  $\widehat{HF}(\Sigma(L))$ , Khovanov’s invariants fit into a long exact sequence relating the invariant for a link and its two resolutions:

$$(2) \quad \dots \longrightarrow \text{Kh}(r(L_0)) \longrightarrow \text{Kh}(r(L_1)) \longrightarrow \text{Kh}(r(L)) \longrightarrow \dots,$$

where here  $r(L)$  denotes the mirror of  $L$ . (Note that our conventions on  $L_0$  and  $L_1$  are opposite to Khovanov’s; this is why we use the mirror.) But unlike  $\widehat{HF}(\Sigma(L))$ , Khovanov’s theory comes with extra gradings (which the maps in the exact sequence respect), which allow one to extract the Jones polynomial from the Betti numbers. (Note that we are using Khovanov’s homology, rather than cohomology, and hence our conventions on  $L_0$  and  $L_1$  are opposite to those of [11].)

The connection between the two link invariants is provided by the following result.

**Theorem 1.1.** *Let  $L \subset S^3$  be a link. There is a spectral sequence whose  $E^2$  term consists of Khovanov’s reduced homology of the mirror of  $L$  with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ , and which converges to  $\widehat{HF}(\Sigma(L); \mathbb{Z}/2\mathbb{Z})$ .*

See Section 6 for a precise statement, and also the proof. Note that in the above statement, we use here a “reduced” version of Khovanov’s homology, which he introduced in [12], with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ .

We have the following quick corollary (whose proof is spelled out in Section 6):

**Corollary 1.2.** *Let  $L \subset S^3$  be a link, and let  $\text{rk}\widetilde{\text{Kh}}(L)$  denote the rank of its reduced Khovanov homology with  $\mathbb{Z}/2\mathbb{Z}$  coefficients. Then, we have the inequality*

$$\det(L) \leq \text{rk}_{\mathbb{Z}/2\mathbb{Z}} \widehat{HF}(\Sigma(L); \mathbb{Z}/2\mathbb{Z}) \leq \text{rk}_{\mathbb{Z}/2\mathbb{Z}} \widetilde{\text{Kh}}(L),$$

where here  $\det(L)$  denotes the determinant of the knot (i.e.  $\text{th}$

Theorem 1.1 is seen as a consequence of a “link surgeries spectral sequence” established in Section 4, which holds in a more general setting (c.f. Theorem 4.1). To place this result in context, recall that if  $K \subset Y$  is a framed knot in a three-manifold, in [14], it is shown that if  $Y_0, Y_1$  denote the result of surgeries on  $Y$  along  $K$  (here, as usual,  $Y_0$  denotes surgery along  $K$  in  $Y$  with respect to the given framing, while  $Y_1$  denotes surgery along  $K$  in  $Y$  with respect to the framing obtained by adding a meridian to the given framing), then there is a long exact sequence relating  $\widehat{HF}(Y)$ ,  $\widehat{HF}(Y_0)$ , and  $\widehat{HF}(Y_1)$ , compare also [6]. When the knot is replaced by a multi-component link, the corresponding object is a spectral sequence relating the various surgeries on the various components of the link. This spectral sequence, in turn, is established with the help of the associativity properties of the pseudo-holomorphic polygon construction, see also [8], [21], [4].

To establish Theorem 1.1 we specialize the link surgeries spectral sequence of Section 4 to the case arising from the branched double cover of a link projection. Given a projection of  $L$ ,  $\Sigma(L)$  comes equipped with a link, whose components correspond to crossings in the projection, framed so that surgeries on these components give branched double-covers of the resolutions of  $L$  (this is the topological input for establishing Equation (1)). With this said, the key observation leading to Theorem 1.1 is the following. Consider the the branched double cover of a collection of unlinks in the plane, connected by cobordisms induced from the connected sums among (and within) the circles. Applying  $\widehat{HF}$  (with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ ) to these objects and morphisms (as required in the  $E^1$  term coming from the link surgery spectral sequence), one recaptures the  $(1+1)$ -dimensional topological quantum field theory which underpins Khovanov’s invariants. Armed with this observation, Theorem 1.1 follows quickly.

**1.1. Further remarks and speculation.** It is interesting to note that the results from Section 3 on non-split alternating links, can be interpreted as saying that the spectral sequence of Theorem 1.1 (c.f. Theorem 6.3 for the precise statement) collapses at the  $E^2$  stage. (Note that it is clear from the more precise statement that if  $L$  is an  $n$ -crossing link, then the spectral sequence always collapses after the  $E^n$  stage.)

A rather striking example where there are non-trivial differentials beyond the  $E^2$  stage is illustrated for torus knots. For example, let  $T_{p,q}$  denotes the  $(p, q)$  torus knot. When both  $p$  and  $q$  are odd,  $\Sigma(T_{p,q})$  is the Brieskorn homology sphere with multiplicities 2,  $p$ , and  $q$ . In particular,  $\Sigma(T_{3,5})$  is the Poincaré homology sphere, so  $\widehat{HF}(\Sigma(T_{3,5}; \mathbb{Z}/2\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}$  (c.f. [18] or [19]), while its reduced Khovanov homology clearly has larger rank, as its Jones polynomial has three non-zero coefficients.

Results from this paper raise a number of further questions, which further link Khovanov's essentially combinatorial theory with problems involving holomorphic disks.

As a first point, observe that the link surgeries spectral sequence works with  $\mathbb{Z}$ , rather than  $\mathbb{Z}/2\mathbb{Z}$  coefficients. This suggests an  $E^2$  term whose  $\mathbb{Z}/2\mathbb{Z}$  reduction agrees with Khovanov's reduced theory, but which differs from the sign conventions as defined by Khovanov. It would be interesting to pin down the new sign conventions of this theory, not only from the point of view of applications to Heegaard Floer homology (i.e. to give information about  $\widehat{HF}$  over  $\mathbb{Z}$  of the branched cover), but also from the point of view of Khovanov's theory, as it would give a link invariant with  $\mathbb{Z}$ -coefficients whose Euler characteristic is the normalized Jones polynomial.

In another direction, it is reasonable to expect that the induced filtered quasi-isomorphism type associated to the branched double cover spectral sequence from Theorem 1.1 is also a link invariant, i.e. that is independent of the projection used in its definition. This would give a countable sequence of link invariants, starting with Khovanov's homology, and ending with  $\widehat{HF}$  of the branched double cover.

This also raises the question of finding a combinatorial description of the higher differentials for the spectral sequence. Although finding a combinatorial description of the Heegaard Floer homology in general is a very interesting, if difficult problem, it is perhaps easier when one specializes to the case of branched double covers of links in the three-sphere.

Another question concerns naturality properties of Khovanov's homology. On the one hand, Khovanov [11] shows that a knot cobordism  $X$  from  $L_1$  to  $L_2$  induces a (combinatorially defined) map between Khovanov homologies. Now, the branched double-cover of  $X$  inside  $[1, 2] \times S^3$  is a four-manifold  $\Sigma(X)$  which gives a cobordism from  $\Sigma(L_1)$  to  $\Sigma(L_2)$ , and correspondingly induces a map on  $\widehat{HF}$  (c.f. [15]), defined by counting holomorphic triangles. This map, in general, can be quite difficult to compute. It is reasonable to expect that there is a well-defined map between the filtered complexes which give rise to Theorem 1.1 and hence between spectral sequences which at the  $E^2$  stage induces Khovanov's map, and at the  $E^\infty$  stage induces the map on  $\widehat{HF}$  induced by  $\Sigma(X)$ .

**1.2. Organization.** The skein exact sequence for  $\widehat{HF}(\Sigma(L))$  is established in Section 2; the results for alternating links (with a sample calculation) are explained in Section 3. The link surgeries spectral sequence is established in Section 4 (note that this general

result applies not only to branched double covers, considered in the rest of the paper). In Section 5, we review Khovanov's link invariant (with  $\mathbb{Z}/2\mathbb{Z}$  coefficients), setting up the notation for Section 6, where we establish the precise form of Theorem 1.1.

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## 2. SKEIN MOVES AND BRANCHED DOUBLE COVERS

Let  $K$  be a framed knot in a three-manifold  $Y$  (i.e. a knot with a choice of longitude  $\lambda$ ). Let  $Y_0 = Y_0(K)$  denote the three-manifold obtained from  $\lambda$ -framed surgery on  $Y$  along  $K$ , and let  $Y_1 = Y_1(K)$  denote the three-manifold obtained from  $\lambda + \mu$ -framed surgery on  $Y$  along  $K$  (where here  $\mu$  denotes the canonical meridian for the knot  $K$ ). We call the ordered triple  $(Y, Y_0, Y_1)$  a *triad* of three-manifolds.

This relationship between  $Y$ ,  $Y_0$ , and  $Y_1$  is symmetric under a cyclic permutation of the three three-manifolds. Indeed, it is not difficult to see that  $(Y, Y_0, Y_1)$  fit into a triad if and only if there is a single oriented three-manifold  $M$  with torus boundary, and three simple, closed curves  $\gamma$ ,  $\gamma_0$ , and  $\gamma_1$  in  $\partial M$  with

$$(3) \quad \#(\gamma \cap \gamma_0) = \#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma) = -1$$

(where here the algebraic intersection number is calculated in  $\partial M$ , oriented as the boundary of  $M$ ), so that  $Y$  resp.  $Y_0$  resp.  $Y_1$  are obtained from  $M$  by attaching a solid torus along the boundary with meridian  $\gamma$  resp.  $\gamma_0$  resp.  $\gamma_1$ .

In [14], we established a long exact sequence connecting  $\widehat{HF}$  for any three three-manifolds which fit into a triad:

$$\dots \longrightarrow \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y_0) \longrightarrow \widehat{HF}(Y_1) \longrightarrow \dots$$

The skein exact sequence for  $\widehat{HF}(\Sigma(L))$  (Equation (1)) follows readily:

**Proposition 2.1.** *Fix a crossing for a projection of a link  $L \subset S^3$ , and  $L_0$  and  $L_1$  be the two resolutions of that crossing as in Figure 1. Then, the three-manifolds  $(\Sigma(L), \Sigma(L_0), \Sigma(L_1))$  form a triad. In particular, there is an induced long exact sequence*

$$\dots \longrightarrow \widehat{HF}(\Sigma(L_0)) \longrightarrow \widehat{HF}(\Sigma(L_1)) \longrightarrow \widehat{HF}(\Sigma(L)) \longrightarrow \dots$$

**Proof.** Fix a sphere  $S$  meeting the link  $L$  in four points, containing a ball  $B$  which contains two arcs of  $L$ , and in whose complement  $L$ ,  $L_0$ , and  $L_1$  agree. Clearly, letting  $M$  be the branched double-cover of  $S^3 - B$  branched along  $L - (L \cap B)$ , we see that  $\Sigma(L)$ ,  $\Sigma(L_0)$ , and  $\Sigma(L_1)$  are all obtained from  $M$  by attaching the branched double cover of  $B$  branched along two arcs.

Now, it is easy to see that the branched double-cover of  $B$  branched along two standard, unknotted arcs is a solid torus. Indeed, a meridian for this solid torus can be realized by pushing either of the two arcs out to the boundary, and taking its branched double-cover. Thus, letting  $\gamma$ ,  $\gamma_0$ , and  $\gamma_1$  denote curves obtained by pushing arcs out into the boundary torus for the  $L$  and its resolutions  $L_0$  and  $L_1$ , it is straightforward to verify that these curves satisfy Equation (3).

Thus,  $(\Sigma(L), \Sigma(L_0), \Sigma(L_1))$  form a triad of three-manifolds. The exact sequence now is a direct consequence of the aforementioned surgery long exact sequence (Theorem 9.12 of [14]; see Theorem 4.7 below for another proof).  $\square$

In particular, we have seen that  $\Sigma(L)$  is obtained as surgery on a knot in  $\Sigma(L_1)$ . This knot can be explicitly seen as the branched double cover of a standard arc inside the three-ball  $B$  containing the two resolved arcs in  $L_1$ . In turn, this arc can be pictured in a knot projection of  $L_1$  as an arc  $A$  which meets  $L_1$  in exactly two points, both of which are on the boundary of  $A$ , and which connect the two resolved strands in  $L_1$ , as pictured in Figure 2.



FIGURE 2. **Obtaining  $\Sigma(L)$  from  $\Sigma(L_1)$ .** The three-manifold  $\Sigma(L)$  (the corresponding to the branched double cover of a link with the a crossing as illustrated on the right) is obtained from  $\Sigma(L_1)$  by surgery on the knot obtained as a branched double cover of the dashed arc indicated in the picture on the left.

## 3. ALTERNATING LINKS

Let  $Y$  be an oriented three-manifold. Let  $|H^2(Y; \mathbb{Z})|$  denote the number of elements in  $H^2(Y; \mathbb{Z})$  provided that  $b_1(Y) = 0$ , and let  $|H^2(Y; \mathbb{Z})| = 0$  if  $b_1(Y) > 0$ . Now, if  $L$  is a link in  $S^3$ , the determinant of  $L$  is defined by  $\det(L) = |\Delta_L(-1)|$ , where here  $\Delta_L(T)$  denotes the Alexander polynomial of  $L$ . It is well-known (see for example [13]) that  $\det(L) = |H^2(\Sigma(L); \mathbb{Z})|$ .

Recall that the Euler characteristic of  $\widehat{HF}(Y)$  is given by  $|H^2(Y; \mathbb{Z})|$  (c.f. Proposition 5.1 of [14]); in particular,  $|H^2(Y; \mathbb{Z})| \leq \text{rk} \widehat{HF}(Y)$ . Three-manifolds with  $b_1(Y) = 0$  for which  $|H^2(Y; \mathbb{Z})| = \text{rk} \widehat{HF}(Y)$  are called  $L$ -spaces (c.f. [16]). This special class of three-manifolds is closed under connected sums and includes all lens spaces and, more generally, all Seifert fibered spaces with finite fundamental group; other examples are given in [19] and [16]. We will prove that if  $L$  is a non-split, alternating link, then  $\Sigma(L)$  is an  $L$ -space. Indeed, the class of links we work with here is wider. To this end, we have the following:

**Definition 3.1.** *The set  $\mathcal{Q}$  of quasi-alternating links is the smallest set of links which satisfies the following properties:*

- (1) *the unknot is in  $\mathcal{Q}$*
- (2) *the set  $\mathcal{Q}$  is closed under the following operation. Suppose  $L$  is any link which admits a projection with a crossing with the following properties:*
  - *both resolutions  $L_0, L_1 \in \mathcal{Q}$ ,*
  - *$\det(L_0), \det(L_1) \neq 0$ ,*
  - *$\det(L) = \det(L_0) + \det(L_1)$ ;**then  $L \in \mathcal{Q}$ .*

Note that quasi-alternating in this sense is different from the notion of almost-alternating, which appears in the literature (c.f. [1]).

**Lemma 3.2.** *Every link which admits a connected, alternating projection is quasi-alternating.*

**Proof.** Recall that a complement of a knot projection in the plane admits a checkerboard coloring. The collection of black regions can be given the structure of a planar graph  $\mathcal{B}(L)$ , whose vertices correspond to black regions and edges correspond to vertices which are corners of pairs of black regions. It is a classical result [3] that if  $L$  admits an alternating projection, then the determinant of  $L$  is the total number of maximal subtrees of the black graph of  $L$ . To fix orientation conventions, when coloring an alternating link, we always use the coloring scheme indicated in Figure 3. (Although this choice is irrelevant to the present discussion, it becomes vital in pinning down signs in our later statements, especially in Theorem 3.4 and Equation (??) below.)



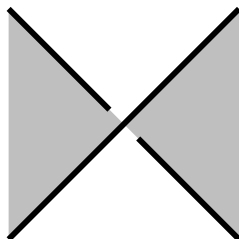


FIGURE 3. **Coloring conventions for alternating knots.** We adopt the pictured convention when coloring an alternating projection.

We now induct on the determinant of the link. In the basic case where the determinant is one, it follows at once that there is only one maximal subtree, and hence that the knot is the unknot.

For the inductive step, it is easy to see that for a reduced alternating projection of  $L$ , if we choose any crossing  $x$ , both resolutions  $L_0$  and  $L_1$  at  $x$  are connected, alternating projections of links. Moreover, it is easy to see that  $\det(L) = \det(L_0) + \det(L_1)$ : maximal subtrees of the black graph of  $L$  which contain, resp. do not contain, the edge corresponding to  $x$  are in one-to-one correspondence with the maximal subtrees of the black graph of  $L_i$ , resp  $L_j$ , where here  $i, j \in \{0, 1\}$  and  $i \neq j$ . Thus, by the inductive hypothesis, the theorem has been established for both  $L_0$  and  $L_1$ ; and hence, the inductive step follows.  $\square$

Of course, there are quasi-alternating links which are not alternating. For a picture of one, see Figure 4.

**Proposition 3.3.** *If  $L$  is a quasi-alternating link,  $\Sigma(L)$  is an  $L$ -space, i.e.*

$$\widehat{HF}(\Sigma(L)) \cong \mathbb{Z}^{\det(L)}.$$

**Proof.** The proposition is now established by induction on the determinant of  $L$ . In the basic case where the determinant is one, it follows at once that there is only one maximal subtree, and hence that the knot is the unknot, so  $\widehat{HF}(\Sigma(L)) = \widehat{HF}(S^3) \cong \mathbb{Z}$ .

The bound  $\det(L) \leq \text{rk} \widehat{HF}(\Sigma(L))$  combined with the long exact sequence Proposition 2.1 readily provides the inductive step (c.f. Proposition 2.1 of [16]).  $\square$

When  $L$  is quasi-alternating, it is not difficult to calculate the absolute gradings of  $\widehat{HF}(\Sigma(L))$ . To this end, we find it convenient to restrict attention to the case where  $L$  is a non-split, alternating link.

Let  $L$  be a link with a connected, alternating projection, and choose a maximal subtree  $T$  of the black graph  $\mathcal{B}(L)$ , and let  $\{e_i\}_{i=1}^m$  denote the edges in  $Z_T = \mathcal{B}(L) - T$ . Let  $V$  denote the lattice generated by these edges. We can equip  $V$  with a bilinear form

$$Q: V \otimes V \longrightarrow \mathbb{Z}$$

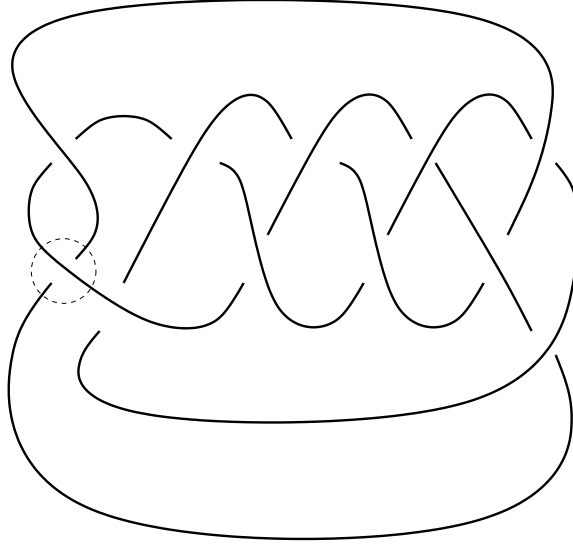


FIGURE 4. **A quasi-alternating, but not alternating, knot.** The pictured knot  $9_{47}$  is quasi alternating: its determinant is 29, and if we resolve the indicated crossing either way, we obtain (non-split) alternating links with determinants 5 and 24.

as follows. Choose orientations for each edge  $e_i \in Z_T$ , let  $C_i$  denote the oriented circuit in  $T \cup \{e_i\}$ ; and if  $X$  is any subgraph of  $\mathcal{B}(L)$ , let  $E(X)$  denote the number of edges in  $X$ . Note that the orientation on  $e_i$  induces an orientation on the circuit  $C_i$ . Given a pair of distinct edges  $e_i, e_j \in Z_T$  with the property that  $C_i \cap C_j \neq \emptyset$ , we let

$$Q(e_i \otimes e_j) = \epsilon(i, j) \cdot E(C_i \cap C_j),$$

where here  $\epsilon(i, j)$  is given by

$$\epsilon(i, j) = \begin{cases} +1 & \text{if the orientation on } C_i \cap C_j \text{ induced from } C_i \\ & \text{is opposite to the one induced from } C_j; \\ -1 & \text{otherwise.} \end{cases}$$

In particular,  $Q(e_i \otimes e_i) = -E(C_i)$ .

We will realize  $Q$  as the intersection form of a four-manifold which bounds  $\Sigma(L)$  in the proof of Theorem 3.4 below.

A characteristic vector for  $M$  is a vector in the lattice  $K \in V^*$  with  $\langle K, v \rangle + Q(v, v) \equiv 0 \pmod{2}$  for each  $v \in V$ . Two characteristic vectors  $K$  and  $K'$  are said to be equivalent if  $K - K' = 2Q(v \otimes \cdot)$  for some  $v \in V$ .

**Theorem 3.4.** *There is an identification of equivalence classes of characteristic vectors for  $Q$  with  $\text{Spin}^c$  structures over  $\Sigma(L)$ . Moreover, given an equivalence class of*

characteristic vectors  $\Xi$ ,  $\widehat{HF}(\Sigma(L), i(\Xi)) \cong \mathbb{Z}$  is supported in dimension

$$d(\Xi) = \frac{\max_{K \in \Xi} K^2 + b}{4},$$

where here  $K^2$  is the length of  $K$  with respect to the inner product on  $V^*$  induced from  $Q$ , and  $b$  is the number of edges in  $Z_T$  (or, more invariantly, the rank of  $H_1(\mathcal{B}(L); \mathbb{Z})$ ).

**Remark 3.5.** We emphasize that we are using the coloring conventions pictured in Figure 3, which breaks the apparent symmetry between the “white” and “black” graphs. In fact, using the white graph in place of the black graph to construct the form analogous to  $Q$ , it is not difficult to see that we obtain the dimensions of the generators for  $\widehat{HF}(-\Sigma(L))$ , whose sign is opposite to those for  $\widehat{HF}(\Sigma(L))$ .

We break the proof into several pieces. First, we construct a four-manifold  $X_L$  which bounds  $\Sigma(L)$ . To construct  $X_L$ , fix a projection for the link  $L$ , and let  $n$  denote its number of crossings. If we form 1-resolutions at each intersection, we obtain a  $k$ -component unlink. The branched double cover of this manifold is  $Y_0 = \#^{k-1}(S^2 \times S^1)$ . Attaching one two-handle for each crossing to “unresolve” the crossing (as in Proposition 2.1), we obtain a cobordism from  $Y_0$  to  $\Sigma(L)$ . Indeed, by filling  $Y_0$  by the boundary connected sum of  $k-1$  copies of  $B^3 \times S^1$ , we obtain a four-manifold  $X_L$  which bounds  $Y$ .

**Lemma 3.6.** *If  $L$  is a non-split alternating link, the four-manifold  $X_L$  described above is negative-definite. Indeed, there is an identification of the form  $Q$  on the vector space  $V$  described above with the intersection form on the two-dimensional homology of  $X_L$ .*

**Proof.** As we have described it,  $X_L$  is built from one zero-handle,  $k-1$  one-handles, and  $n$  two-handles. In fact, the tree  $T$  specifies  $k-1$  two-handles which cancel the one-handles; i.e. after attaching the two-handles from the tree, we obtain the branched double cover of a single unknot, which is  $S^3$ . Now,  $X_L$  is obtained from the four-ball by surgery on a link in  $S^3$  (the branched double cover of the unknot) whose components correspond to the remaining edges in  $\mathcal{B}(L) - T$  (i.e. the link components are the branched double covers of the arcs with boundary in the unknot, which are associated to the edges in  $\mathcal{B}(L) - T$ ). We claim that a choice of orientation on each edge  $e_i$  simultaneously orients all the components of this link, up to an overall sign.

To see this, we proceed as follows. Let  $U$  denote the unknot as specified by the tree  $T$ . Let  $\phi$  be a vector field normal to  $U$  which is orthogonal to the kernel of the projection map used in describing the knot projection. This vector field  $\phi$ , of course, specifies the blackboard framing of  $U$ . The vector field  $\phi$  has two possible lifts in the branched double cover of the unknot (in the sense that there are two lifts in the branched double cover of the knot obtained by displacing  $U$  by  $\phi$ ). Choose one, and denote it  $\tilde{\phi}$  (while the other is denoted  $\tilde{\phi}'$ ). Then, the knot corresponding to  $e_i$  (thought of as an arc

connecting  $x$  to  $y$  in the unknot) is oriented so that its tangent vector at  $x$  agrees with  $\tilde{\phi}_x$  (as opposed to  $\tilde{\phi}'_x$ ). We denote the oriented knot associated to  $e_i$  with its orientation by  $k_i$ . (Note that the other lift of the blackboard framing has the effect of reversing the induced orientations on all the knots  $k_i$  simultaneously.)

Next, we argue that the intersection form of  $X_L$  is negative-definite. We prove this by induction on the number of crossings. The basic case is obvious. Next, recall that

$$\left| H^2(\Sigma(L); \mathbb{Z}) \right| = \left| H^2(\Sigma(L_0); \mathbb{Z}) \right| + \left| H^2(\Sigma(L_1); \mathbb{Z}) \right|,$$

so it readily follows that the two-handle from  $\Sigma(L_1)$  to  $\Sigma(L)$  (and also the one from  $\Sigma(L)$  to  $\Sigma(L_0)$ ) is negative-definite. Now, it is easily seen that  $X_L$  is obtained by attaching this negative-definite two-handle to  $X_{L_1}$ .

We show that the intersection form on  $H_2(X_L; \mathbb{Z})$  is given by  $Q$ . To this end, observe that if we attach  $m$  of the remaining two handles  $\{e_i\}_{i=1}^m$ , to  $S^3$ , the number of elements in  $H^2$  of the boundary three-manifold (with  $\mathbb{Z}$  coefficients) is given by the determinant of the matrix  $(Q(e_i \otimes e_j))_{i,j \in \{1, \dots, n\}}$ , which in turn is obtained from the number of maximal subtrees in  $T \cup \{e_i\}_{i=1, \dots, m}$ .

In particular, since  $X_L$  is negative-definite, it follows at once that if we choose a basis for  $H_2(X_L; \mathbb{Z})$  given by the two-handles in  $X_L$  (with any set of orientations), then if  $[e_i]$  the homology class corresponding to an edge  $e_i$ , then  $\#[e_i] \cap [e_i] = -E(C_i)$  (since the number of maximal subtrees of a circuit is the length of the circuit, and the sign is forced by the negative-definiteness).

When  $i \neq j$ ,  $\#[e_i] \cap [e_j]$  is given by the linking number of  $k_i$  with  $k_j$ . This in turn is calculated in a model case: consider the unknot corresponding to the tree  $T$ , together with the two arcs corresponding  $e_i$  and  $e_j$ . This is easily seen to be isotopic to an unknot with two arcs attached, in a manner which has a standard projection depending only on the integer  $E(C_i \cap C_j)$ , as pictured in Figure 5. Unwinding the unknot and taking the branched double cover, we see that the branched double covers of the original arcs become circles which are linked  $|E(C_i \cap C_j)|$  times. It is then straightforward to see that the sign of this linking number is the one stated (once we choose a lift of the blackboard framing for the unknot).  $\square$

**Proof of Theorem 3.4.** With Lemma 3.6, the proof of the theorem now follows along the lines of Section 2 of [19]. We sketch here the main points. Let  $\text{Ch}(X_L)$  denote the set of characteristic vectors for the intersection form  $H^2(X_L; \mathbb{Z})$ . We write  $K \sim K'$  if there is an element  $v \in H^2(X_L, \partial X_L)$  with the property that  $K = K' + 2v$ . Next (compare [19]), consider the subgroup

$$\widehat{\mathbb{H}}(X_L) \subset \text{Hom}(\text{Ch}(X_L), \mathbb{Z}),$$

consisting of maps  $\phi$  with the properties that

- $\phi(K) = \phi(K')$  if  $K \sim K'$  and  $Q(K, K) = Q(K', K')$

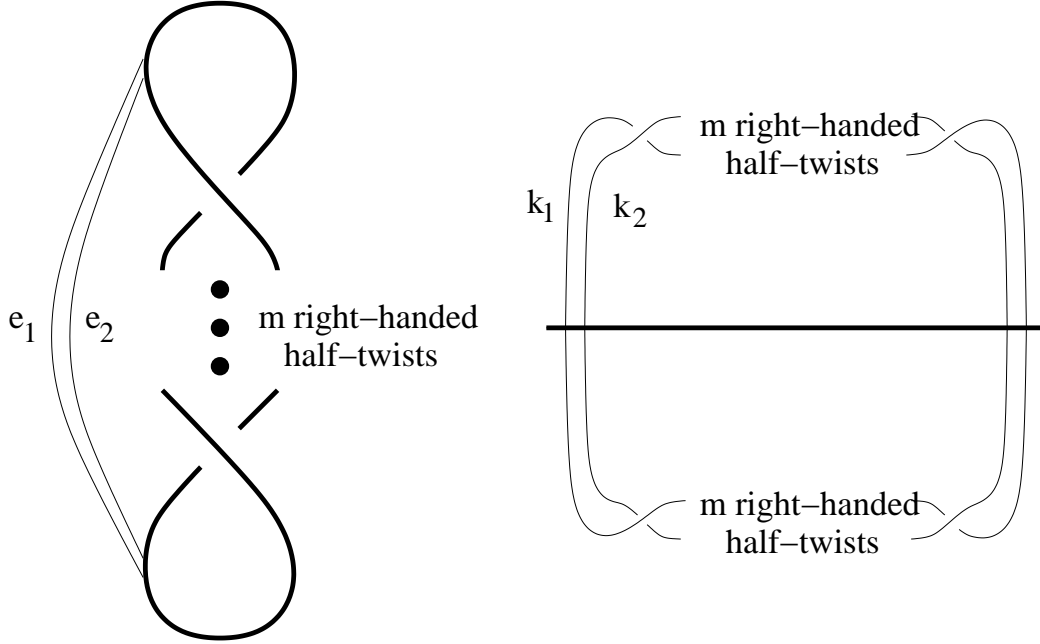


FIGURE 5. **Standard picture for two crossings.** If  $e_1$  and  $e_2$  correspond to two closed circuits with  $m = E(C_1 \cap C_2)$  edges in common, then the unknot corresponding to the tree  $T$ , together with the two arcs associated to  $e_1$  and  $e_2$ , is isotopic to the picture on the left (which depicts a projection of the unknot with  $m$  right-handed half-twists in it, of which two have already been drawn, together with two arcs which meet the unknot in the specified manner). Passing to the branched double cover of the unknot (which in turn is best visualized by unwinding the twists on the unknot, at the expense of twisting the arcs corresponding to  $e_1$  and  $e_2$ , and then taking the branched double cover of these edges to obtain knots  $k_1$  and  $k_2$ , we obtain the picture shown on the right, where the solid line indicates the branched locus.

- $\phi(K) = 0$  if there is some  $K' \sim K$  with  $Q(K', K') > Q(K, K)$ .

Viewing  $X_L$  as a cobordism from  $-\Sigma(L)$  to  $S^3$ , we obtain a naturally induced map (c.f. [15])

$$T_{X_L}: \widehat{HF}(-\Sigma(L)) \longrightarrow \text{Hom}(\text{Ch}(X_L), \mathbb{Z}),$$

in view of the fact that  $\widehat{HF}(S^3) \cong \mathbb{Z}$ .

Unless the diagram of  $L$  represents the unknot, we can always find a double-point  $p$  whose two resolutions are connected diagrams. This gives the following commutative

diagram:

$$\begin{array}{ccccccc}
(4) & 0 & \longrightarrow & \widehat{HF}(-\Sigma(L_0)) & \longrightarrow & \widehat{HF}(-\Sigma(L)) & \longrightarrow & \widehat{HF}(-\Sigma(L_1)) & \longrightarrow & 0 \\
& & & \downarrow T_{X_{L_0} \# \overline{\mathbb{CP}}^2} & & \downarrow T_L & & \downarrow T_{L_1} & & \\
& & & \text{Hom}(\text{Ch}(X_{L_0} \# \overline{\mathbb{CP}}^2), \mathbb{Z}) & \xrightarrow{\mathbb{A}} & \text{Hom}(\text{Ch}(X_L), \mathbb{Z}) & \xrightarrow{\mathbb{B}} & \text{Hom}(\text{Ch}(X_{L_1}), \mathbb{Z}) & & 
\end{array}$$

where the top row is exact, the squares commute, the maps  $\mathbb{A}$  and  $\mathbb{B}$  are given by

$$\begin{aligned}
\mathbb{A}(\phi_0)(K) &= \sum_{\{K_0 \in \text{Ch}(X_{L_0} \# \overline{\mathbb{CP}}^2) \mid K_0|_{H^2(X_L; \mathbb{Z})} = K\}} \phi_0(K_0) \\
\mathbb{B}(\phi)(K) &= \sum_{\{K_1 \in \text{Ch}(X_{L_1}) \mid K_1|_{H^2(X_{L_1}; \mathbb{Z})} = K\}} \phi(K_1).
\end{aligned}$$

A straightforward induction on the number of crossings in the diagram shows that the image of  $T_{X_L}$  is contained in  $\widehat{\mathbb{H}}(X_L)$ . The sphere with square  $-1$  contained in the composite cobordism from  $\Sigma(L_0)$  to  $\Sigma(L_1)$  through  $\Sigma(L)$  is used to show that  $\mathbb{B} \circ \mathbb{A} = 0$ , and also that  $\mathbb{A}$  is injective. Straightforward homological algebra then shows that  $T_L$  is an isomorphism, again, by induction on the number of crossings, together with Diagram 4, and an identification  $\widehat{\mathbb{H}}(X_{L_0}) \cong \widehat{\mathbb{H}}(X_{L_0} \# \overline{\mathbb{CP}}^2)$ . (For a more detailed argument establishing an analogous result, see the proof of Lemma 2.10 of [19].)

Endow  $\widehat{\mathbb{H}}(X_L)$  with a grading, by declaring an element to be homogeneous of degree  $d$  if it supported on those  $K \in \text{Ch}(X_L)$  with

$$- \left( \frac{K^2 + \text{rk} H^2(X_L)}{4} \right) = d.$$

Clearly,  $T_{X_L}$  carries  $\widehat{HF}_d(-\Sigma(L))$  to  $\widehat{\mathbb{H}}_d(X_L)$  (c.f. [18]). Since  $\widehat{HF}_d(-\Sigma(L)) \cong \widehat{HF}^{-d}(\Sigma(L))$  (c.f. [18]), the result now follows.  $\square$

Note that the long exact sequence can be pushed slightly further than we have done in the above discussion. For example, recall that if  $W$  is a cobordism between two spaces  $L$ -spaces with  $b_2^+ > 0$ , then the induced map on  $HF^+$  is trivial (c.f. [15]). This gives at once the result that if  $L$  differs from an alternating (or indeed quasi-alternating) knot by at a single crossing, then all the elements of  $HF_{\text{red}}^+(\Sigma(L))$  have the same  $\mathbb{Z}/2\mathbb{Z}$ -grading. Indeed, the map induced by a two-handle from  $\Sigma(L_0)$  to  $\Sigma(L_1)$ , where  $L_0$  and  $L_1$  are both quasi-alternating, determined purely by homological information. This can be used to give information about the Heegaard Floer homology of  $\Sigma(L)$  when its two resolutions are quasi-alternating. We do not pursue this any further here, contenting ourselves instead with a sample calculation illustrating Theorem 3.4.

3.1. **An example:**  $9_{40}$ . To illustrate Theorem 3.4, we calculate  $\widehat{HF}(\Sigma(L))$  where  $L$  is the alternating knot with nine crossings  $9_{40}$ , pictured in Figure 6.

The black graph of this knot is illustrated in Figure 7. Using as our base tree  $T$  the solid edges pictured in the figure, and the orientations of the remaining edges indicated, the intersection form of  $X_L$  takes the form

$$Q_L = \begin{pmatrix} -3 & -2 & -1 & -1 \\ -2 & -5 & -2 & -3 \\ -1 & -2 & -4 & -3 \\ -1 & -3 & -3 & -5 \end{pmatrix}.$$

It is a straightforward if tedious matter to find the maximal lengths of the characteristic vectors for  $Q$  in its equivalence classes. Note that this is a finite search: it is easy to see that all maximal characteristic vectors have the property that  $|\langle K, v \rangle| \leq |Q(v \otimes v)|$ , and hence to determine the absolute gradings of the generators of  $\widehat{HF}(\Sigma(L))$  (this and the further calculations in this section were all done with the help of Mathematica [22]). We display the results below. The numbers are ordered as suggested by the group structure of  $H^2(\Sigma(L); \mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$ ; i.e. having chosen such an isomorphism, we have a naturally induced identification  $\text{Spin}^c(\Sigma(L)) \cong \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$  (where we choose as

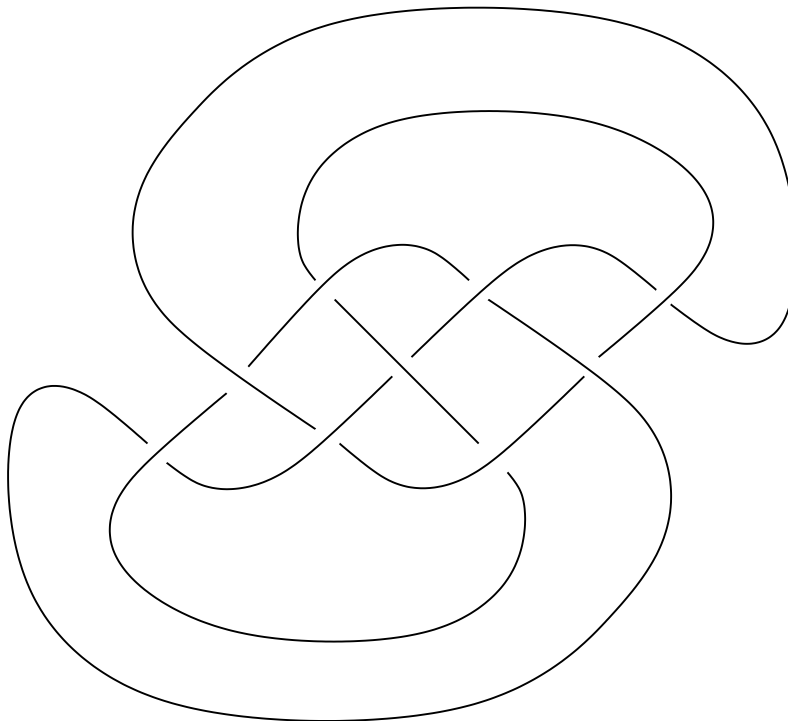


FIGURE 6. The knot  $9_{40}$ .

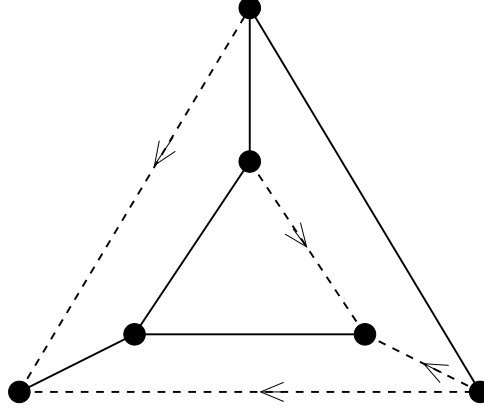


FIGURE 7. **Black graph for the knot  $9_{40}$ .** All edges (including those which are dashed) are included in the graph. The solid edges constitute the tree  $T$  used for the matrix given in the text. The dashed edges, when oriented, give rise to the matrix in the text.

the origin the spin structure on  $\Sigma(L)$ ; since  $H_1$  has no two-torsion, this structure is uniquely determined); i.e. the element in the  $i^{th}$  row (counting from 0 to 4) and  $j^{th}$  column (counting from 0 to 14) is the absolute grading of the element in the  $\text{Spin}^c$  structure corresponding to  $(i, j) \in \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$ .

$-\frac{1}{2}$	$\frac{13}{30}$	$-\frac{23}{30}$	$-\frac{9}{10}$	$-\frac{11}{30}$	$\frac{1}{30}$	$-\frac{1}{10}$	$\frac{1}{30}$	$\frac{1}{30}$	$-\frac{1}{10}$	$-\frac{11}{30}$	$-\frac{23}{30}$	$-\frac{9}{10}$	$\frac{13}{30}$	$-\frac{11}{30}$
$\frac{7}{10}$	$-\frac{11}{30}$	$\frac{13}{30}$	$-\frac{1}{10}$	$\frac{13}{30}$	$\frac{5}{6}$	$\frac{3}{10}$	$\frac{13}{30}$	$\frac{13}{30}$	$-\frac{1}{10}$	$-\frac{11}{30}$	$-\frac{23}{30}$	$\frac{7}{10}$	$\frac{1}{30}$	$-\frac{23}{30}$
$\frac{3}{10}$	$-\frac{23}{30}$	$\frac{1}{30}$	$-\frac{9}{10}$	$-\frac{11}{30}$	$\frac{1}{30}$	$-\frac{9}{10}$	$-\frac{23}{30}$	$-\frac{23}{30}$	$\frac{3}{10}$	$\frac{1}{30}$	$-\frac{11}{30}$	$\frac{7}{10}$	$\frac{1}{30}$	$-\frac{23}{30}$
$\frac{3}{10}$	$-\frac{23}{30}$	$\frac{1}{30}$	$\frac{7}{10}$	$-\frac{23}{30}$	$-\frac{11}{30}$	$\frac{3}{10}$	$\frac{13}{30}$	$\frac{13}{30}$	$-\frac{9}{10}$	$\frac{5}{6}$	$\frac{13}{30}$	$-\frac{9}{10}$	$\frac{13}{30}$	$-\frac{11}{30}$
$\frac{7}{10}$	$-\frac{11}{30}$	$\frac{13}{30}$	$\frac{7}{10}$	$-\frac{23}{30}$	$-\frac{11}{30}$	$-\frac{1}{10}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{3}{10}$	$\frac{1}{30}$	$-\frac{11}{30}$	$-\frac{1}{10}$	$-\frac{23}{30}$	$\frac{13}{30}$



## 4. THE LINK SURGERIES SPECTRAL SEQUENCE

In this section, we turn our attention away from branched double-covers, and consider the case of a general three-manifold  $Y$ . Our aim here is to describe a generalization of the surgery long exact sequence for the case of multi-component links in  $Y$ . In the course of making this generalization, we give a quick (and slightly stronger) proof of the long exact sequence based on associativity properties of the holomorphic polygon construction, combined with some homological algebra discussed in Subsection 4.1. But first, we introduce some notation.

Let  $L = K_1 \cup \dots \cup K_\ell$  be an  $\ell$ -component, framed link in a three-manifold  $Y$ . A “multi-framing” is a vector  $I = (m_1, \dots, m_\ell)$ , where each  $m_i \in \{0, 1, \infty\}$ . For a multi-framing, there is a three-manifold  $Y(I)$ , which is obtained from  $Y$  by performing  $m_i$ -framed surgery on the component  $K_i$  for  $i = 1, \dots, \ell$ . As usual, when  $m_i = \infty$ , this means no surgery,  $m_i = 0$  this means  $\lambda_i$ -framed surgery, and when  $m_i = 1$ , this is surgery of type  $\mu_i + \lambda_i$ .

We give the set  $\{0, 1, \infty\}^\ell$  the lexicographical ordering (with the understanding that  $0 < 1 < \infty$ ). If  $I \in \{0, 1, \infty\}^\ell$ , we call  $I'$  an *immediate successor* of  $I$  if  $I = (m_1, \dots, m_\ell)$  and  $I' = (m'_1, \dots, m'_\ell)$  if there is some  $j$  so that for all  $i \neq j$ ,  $m_i = m'_i$ , while  $m_j < m'_j$ , excluding the case where  $m_j = 0$  and  $m'_j = \infty$ . Clearly, if  $I'$  is an immediate successor of  $I$ , there is a corresponding map on Floer homology

$$\widehat{G}_{I < I'} : \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y')$$

associated to the single two-handle addition.

We now state the link surgeries spectral sequence alluded to in the introduction:

**Theorem 4.1.** *There is a spectral sequence whose  $E^1$  term is  $\bigoplus_{I \in \{0,1\}^\ell} \widehat{HF}(Y(I))$ , all of whose differentials respect the filtration given by the dictionary ordering on  $\{0,1\}^\ell$ , and whose  $d_1$  differential is obtained by adding all  $\widehat{G}_{I < I'}$  (where  $I'$  is an immediate successor of  $I$ ).*

**Remark 4.2.** *It is sometimes convenient to pass from “cubically filtered” complexes as described above to  $\mathbb{Z}$ -filtered ones. This procedure – “flattening” the cube (c.f. [11]) – is done, for example, by letting  $E_m^1 = \bigoplus_{\{I \in \{0,1\}^\ell \mid \sum_{i=1}^\ell I_i = m\}} E_I^1$ . Note that the length of this filtration is one greater than the number of components of the link.*

**Remark 4.3.** *Although we have stated Theorem 4.1 for  $\widehat{HF}$ , the same result can be established for  $HF^+$ , with some notational changes.*

Before proceeding to the proof, we indulge in a purely homological-algebraic digression. The algebraic here was inspired by a conversation with Paul Seidel, who communicated to us some version of Lemma 4.4.

**4.1. Mapping cones.** We begin with some terminology.

Let  $A_1$  and  $A_2$  be a pair of chain complexes. A chain map

$$\phi: A_1 \longrightarrow A_2$$

is called a *quasi-isomorphism* if the induced map on homology is an isomorphism. Two chain complexes  $A_1$  and  $A_2$  are said to be *quasi-isomorphic* if there is a third chain complex  $B$  and a pair of quasi-isomorphisms  $\phi_1: A_1 \longrightarrow B$  and  $\phi_2: A_2 \longrightarrow B$ .

Recall that if we have a chain map between chain complexes  $f_1: A_1 \longrightarrow A_2$ , we can form its mapping cone  $M(f_1)$ , whose underlying module is the direct sum  $A_1 \oplus A_2$ , endowed with the differential

$$\partial = \begin{pmatrix} \partial_1 & 0 \\ f_1 & \partial_2 \end{pmatrix},$$

where here  $\partial_i$  denotes the differential for the chain complex  $A_i$ . (When working with coefficients in characteristic different from 2, for the above formula to work, we adopt the convention that for a chain map,  $\partial_2 \circ f_1 + f_1 \circ \partial_1 = 0$ .) Recall that there is a short exact sequence of chain complexes

$$0 \longrightarrow A_2 \xrightarrow{\iota} M(f_1) \xrightarrow{\pi} A_1 \longrightarrow 0.$$

This induces a long exact sequence, for which the connecting homomorphism is the map on homology induced by  $f_1$ .

The mapping cylinder is natural in the following sense. Suppose that we have a diagram of chain complexes

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & A_2 \\ \psi_1 \downarrow & & \downarrow \psi_2 \\ B_1 & \xrightarrow{g_1} & B_2 \end{array}$$

which commutes up to homotopy, then there is an induced map

$$m(\psi_1, \psi_2): M(f_1) \longrightarrow M(g_1)$$

which fits into the following diagram, where the rows are exact and the squares are homotopy-commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_2 & \longrightarrow & M(f_1) & \longrightarrow & A_1 \longrightarrow 0 \\ & & \psi_2 \downarrow & & m(\psi_1, \psi_2) \downarrow & & \downarrow \psi_1 \\ 0 & \longrightarrow & B_2 & \longrightarrow & M(g_1) & \longrightarrow & B_1 \longrightarrow 0. \end{array}$$

**Lemma 4.4.** *Let  $\{A_i\}_{i=1}^\infty$  be a collection of chain maps and let*

$$\{f_i: A_i \longrightarrow A_{i+1}\}_{i \in \mathbb{Z}}$$

*be a collection of chain maps satisfying the following two properties:*

(1)  $f_{i+1} \circ f_i$  is chain homotopically trivial, by a chain homotopy

$$H_i: A_i \longrightarrow A_{i+2}$$

(2) the difference

$$f_{i+2} \circ H_i - H_{i+1} \circ f_i: A_i \longrightarrow A_{i+3}$$

is a quasi-isomorphism.

Then,  $M(f_2)$  is quasi-isomorphic to  $A_4$ .

**Proof.** Hypothesis (1) shows that the maps induced on homology by the  $f_i$  fit together to form a chain complex:

$$(5) \quad \dots H_*(A_1) \xrightarrow{f_1} H_*(A_2) \xrightarrow{f_2} H_*(A_3) \xrightarrow{f_3} H_*(A_4) \dots$$

Letting  $\psi_i = f_{i+2} \circ H_i + H_{i+1} \circ f_i: A_i \longrightarrow A_{i+3}$ , then Hypothesis (1) proves that  $\psi_i$  is a chain map; and indeed that the square

$$(6) \quad \begin{array}{ccc} A_i & \xrightarrow{f_i} & A_{i+1} \\ \psi_i \downarrow & & \downarrow \psi_{i+1} \\ A_{i+3} & \xrightarrow{f_{i+3}} & A_{i+4} \end{array}$$

commutes up to homotopy.

Next, define  $\alpha_i: M(f_i) \longrightarrow A_{i+2}$  by:

$$\alpha_i(a_i, a_{i+1}) = H_i(a_i) - f_{i+1}(a_{i+1}),$$

and  $\beta_i: A_i \longrightarrow M(f_{i+1})$  by

$$\beta_i(a_i) = (f_i(a_i), H_i(a_i)).$$

Now,

$$\alpha_{i+1} \circ \beta_i = \psi_i,$$

which is a quasi-isomorphism.

Moreover, consider the diagram:

$$(7) \quad \begin{array}{ccccccccc} A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{\iota_3} & M(f_2) & \xrightarrow{\pi_3} & A_2 & \xrightarrow{f_2} & A_3 \\ = \downarrow & & = \downarrow & & \alpha_2 \downarrow & & \psi_2 \downarrow & & \psi_3 \downarrow \\ A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 & \xrightarrow{f_5} & A_6 \\ \psi_2 \downarrow & & \psi_3 \downarrow & & \beta_4 \downarrow & & = \downarrow & & = \downarrow \\ A_5 & \xrightarrow{f_5} & A_6 & \xrightarrow{\iota_6} & M(f_5) & \xrightarrow{\pi_5} & A_5 & \xrightarrow{f_5} & A_6 \end{array}$$

The map  $X: M(f_2) \longrightarrow A_5$  defined by

$$X(a_2, a_3) = H_2(a_2) + f_3(a_3)$$

gives a chain homotopy between  $\psi_2 \circ \pi$  and  $f_4 \circ \alpha$ , while the map  $Y: A_3 \longrightarrow M(f_5)$  defined by

$$Y(a_3) = (H_3 a_3, 0)$$

gives a chain homotopy between  $f_3 \circ \beta_4$  and  $\iota \circ \psi_3$ . Thus, all the squares in Diagram (7) commute up to homotopy, and the maps induced on homology on the top and bottom rows are exact. From the five-lemma, it follows that the map induced on homology  $\beta_4 \circ \alpha_2$  is also an isomorphism. Thus (in view of the fact that  $\alpha_5 \circ \beta_4$  is a quasi-isomorphism, we conclude  $\beta_4$  and hence  $\alpha_2$  is a quasi-isomorphism.  $\square$

It is useful to interpret Lemma 4.4 in the following terms. Under the hypotheses of that lemma, we can form an “iterated mapping cone”  $M(f_1, f_2, f_3)$  whose underlying module is  $A_1 \oplus A_2 \oplus A_3$ , and whose differential is given by the matrix

$$(8) \quad \partial = \begin{pmatrix} \partial_1 & 0 & 0 \\ f_1 & \partial_2 & 0 \\ H_1 & f_2 & \partial_3 \end{pmatrix}.$$

Indeed, Hypothesis (1) guarantees that  $\partial$  determines a differential on  $M(f_1, f_2, f_3)$ . Lemma 4.4 can be interpreted as saying that  $H_*(M(f_1, f_2, f_3)) = 0$ . Now, there is a short exact sequence

$$0 \longrightarrow A_3 \longrightarrow M(f_1, f_2, f_3) \longrightarrow M(f_1) \longrightarrow 0,$$

and the connecting homomorphism

$$H_*(M(f_1)) \longrightarrow H_*(A_3)$$

is the map on homology induced by the map  $\alpha_1$  in the above lemma.

**4.2. Pseudo-holomorphic  $n$ -gons.** With the above homological algebra in place, we proceed to the geometrical underpinnings of Theorem 4.1. We will make heavy use of the pseudo-holomorphic polygon construction, c.f. [8], [21], [4] and its relationship with Heegaard Floer homology, as explained in Section 8 of [17]. We recall this construction very briefly here.

Let  $\Sigma$  be a connected, closed, oriented two-manifold of genus  $g$ , and fix  $(m+1)$   $g$ -tuples of attaching circles  $\{\boldsymbol{\eta}^i\}_{i=1}^m$ . Specifically, for each fixed  $i$ , the set  $\boldsymbol{\eta}^i = \{\eta_j^i\}_{j=1}^g$  is a collection of  $g$  pairwise disjoint, homologically linearly independent, embedded curves in  $\Sigma$ . We choose also a reference point  $\Sigma$  disjoint from all the  $\eta_j^i$ . In the terminology of [17], this data  $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^m, z)$  is called a pointed Heegaard  $(m+1)$ -tuple. We have a map of groups

$$\widehat{f}_{\boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^m}: \bigotimes_{i=1}^m \widehat{CF}(Y_{\eta^{i-1}, \eta^i}) \longrightarrow \widehat{CF}(Y_{\eta^0, \eta^m}),$$

where here  $Y_{\eta^i, \eta^j}$  denotes the three-manifold described by the Heegaard diagram  $(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^j)$ . This map is obtained by counting pseudo-holomorphic  $m+1$ -gons in  $\text{Sym}^g(\Sigma)$  which are disjoint from the subvariety  $\{z\} \times \text{Sym}^{g-1}(\Sigma)$ .

More precisely, let  $\mathbb{T}_{\eta^i}$  denote the  $g$ -dimensional torus  $\eta_1^i \times \dots \times \eta_g^i$  in the  $g$ -fold symmetric product  $\text{Sym}^g(\Sigma)$ . A *Whitney*  $(m+1)$ -gon is a map  $u$  from the standard  $(m+1)$ -gon into  $\text{Sym}^g(\Sigma)$  which maps the  $i^{\text{th}}$  edge into  $\mathbb{T}_{\eta^i}$  (where here the edges are labelled  $0, \dots, m$ ). Fixing  $\mathbf{x}_i \in \mathbb{T}_{\eta^{i-1}} \cap \mathbb{T}_{\eta^i}$  and  $\mathbf{y} \in \mathbb{T}_{\eta^0} \cap \mathbb{T}_{\eta^m}$ , we let  $\pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y})$  denote the set of homotopy classes of Whitney  $(m+1)$ -gons which, for  $i = 1, \dots, m$ , map the vertex between the  $(i-1)^{\text{st}}$  and the  $i^{\text{th}}$  edge to  $\mathbf{x}_i$  and the vertex between  $m^{\text{th}}$  and  $0^{\text{th}}$  edges to  $\mathbf{y}$ .

For fixed  $\varphi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y})$ , we let  $\mathcal{M}(\varphi)$  denote the set of pseudo-holomorphic representatives for  $\varphi$ . With this notation in place, then, the map  $\widehat{f}_{\eta^0, \dots, \eta^m}$  (when  $m > 1$ ) is defined by

$$\widehat{f}_{\eta^0, \dots, \eta^m}(\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_m) = \sum_{\mathbf{y} \in \mathbb{T}_{\eta^0} \cap \mathbb{T}_{\eta^m}} \sum_{\{\varphi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}) \mid \mu(\varphi)=0, n_z(\varphi)=0\}} (\#\mathcal{M}(\varphi)) \cdot \mathbf{y},$$

where here  $n_z(\varphi)$  denotes the intersection number of  $\varphi$  with the subvariety  $\{z\} \times \text{Sym}^{g-1}(\Sigma) \subset \text{Sym}^g(\Sigma)$ , and  $\mu(\varphi)$  denotes the expected dimension of the moduli space  $\mathcal{M}(\varphi)$  (i.e. the Maslov index of  $\varphi$ , c.f. [5], [20]). In the special case where  $m = 1$ , we sum over homotopy classes with  $\mu(\varphi) = 1$ , and count points in the quotient space  $\mathcal{M}(\varphi)/\mathbb{R}$ . Thus, when  $m = 1$ , the map  $\widehat{f}_{\eta^0, \eta^1}$  is simply the differential for the Heegaard Floer chain complex for  $Y_{\eta^0, \eta^1}$ , and when  $m = 2$ ,  $\widehat{f}_{\eta^0, \eta^1, \eta^2}$  is the chain map induced from the counts of pseudo-holomorphic triangles.

These maps are well-known to satisfy a generalized associativity property, c.f. [8], [21], [4]:

$$(9) \quad \sum_{0 \leq i < j \leq m} \widehat{f}_{\eta^i, \eta^{i+1}, \dots, \eta^j} \circ \widehat{f}_{\eta^0, \dots, \eta^{i-1}, \eta^i, \eta^j, \dots, \eta^m} = 0.$$

For example, when  $m = 1$ , the above associativity statement is the is equivalent to the fact that the square of the differential for  $\widehat{CF}(Y_{\eta^0, \eta^1})$  is trivial. When  $m = 2$ , associativity asserts that the maps induced by holomorphic triangles are chain maps, and when  $m = 3$ , it states that the triangle pairing is associative, up to chain homotopy (hence the name). Thinking of the tuples  $\boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^m$  as corresponding to edges of an  $(m+1)$ -gon, we see that for any pair of edges, there is a corresponding degeneration of the  $(m+1)$ -gon as a juxtaposition of a pair of  $a$ - and  $b$ -gons, with  $a + b - 3 = m$ . The above sum a composition over all such degenerations of the corresponding composition of maps.

We can construct Heegaard diagrams for the  $Y(I)$  as follows. Given  $Y$  with the framed link  $L$ , we can construct a Heegaard triple  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ , where here  $\beta_1, \dots, \beta_\ell$  are meridians for the links,  $\gamma_1, \dots, \gamma_\ell$  are corresponding framing curves (and  $\gamma_{\ell+1}, \dots, \gamma_g$  are exact Hamiltonian translates of  $\beta_{\ell+1}, \dots, \beta_g$ ). (For more on the construction this diagram, see Section 4 of [15]. In the terminology of that paper, the Heegaard triple we are considering is the Heegaard triple subordinate to some bouquet for the framed

link  $L$ .) We also choose curves  $\delta_1, \dots, \delta_\ell$  to be corresponding curves representing the framings obtained by adding meridians to the original framings (and  $\delta_{\ell+1}, \dots, \delta_g$  are exact Hamiltonian translates of the  $\beta_{\ell+1}, \dots, \delta_g$ ). Given  $I \in \{0, 1, \infty\}^\ell$ , let  $\boldsymbol{\eta}(I) = \{\eta_1, \dots, \eta_g\}$  denote the  $g$ -tuple of attaching circles, where here

$$\eta_i = \begin{cases} \beta_i & \text{if } i > n \text{ or } m_i = \infty \\ \gamma_i & \text{if } m_i = 0 \\ \delta_i & \text{if } m_i = 1. \end{cases}$$

Thus, a Heegaard diagram for  $Y(I)$  is given by  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I), z)$ .

Given a sequence of multi-framings  $I^0 < \dots < I^k$ , there is an induced map

$$D_{I^0 < \dots < I^k} : \widehat{CF}(Y(I^0)) \longrightarrow \widehat{CF}(Y(I^k))$$

defined by

$$(10) \quad D_{I^0 < \dots < I^k}(\xi) = \widehat{f}_{\alpha, \eta(I^0), \dots, \eta(I^k)}(\xi \otimes \widehat{\Theta}_1 \otimes \dots \otimes \widehat{\Theta}_k),$$

where  $\widehat{\Theta}_i$  are cycles representing the canonical top-dimensional generators for  $\widehat{HF}$  of  $Y_{\eta(I^i), \eta(I^{i+1})}$ , which is a connected sum of several copies of  $(S^2 \times S^1)$ . (We will be lax about distinguishing here between intersection points in  $\mathbb{T}_{\eta(I^i)} \cap \mathbb{T}_{\eta(I^{i+1})}$  and generators of the homology groups  $\widehat{HF}(\#^k(S^2 \times S^1))$ ; after all, one can equip this three-manifold with a Heegaard diagram for which the intersection points all correspond to closed chains, and hence each generator is represented by a unique intersection point.)

Let  $X = \bigoplus_{I \in \{0, 1, \infty\}^\ell} \widehat{CF}(Y(I))$ , endowed with the map

$$D : X \longrightarrow X,$$

defined by

$$D\xi = \sum_J \sum_{\{I^1 < \dots < I^k = J\}} D_{I^1 < \dots < I^k}(\xi),$$

where here the index set of the inner sum is the set of all increasing sequences connecting  $I$  to  $J$ , with the property that for all  $i = 1, \dots, k+1$ ,  $I^{i+1}$  is an immediate successor of  $I^i$ .

**Lemma 4.5.** *Fix  $I, J \in \{0, 1, \infty\}^\ell$ . We have that*

$$\sum_{I=I^0 < I^1 < \dots < I^k=J} \widehat{f}_{\eta(I^0), \dots, \eta(I^k)}(\widehat{\Theta}_1 \otimes \dots \otimes \widehat{\Theta}_k) \equiv 0,$$

where again the sum is taken over sequences with the property that  $I^{i+1}$  is an immediate successor of  $I^i$ .

**Proof.** We consider the case where  $k > 2$ . In this case, there is a juxtaposition of triangles representing

$$\widehat{f}_{\eta(I^0), \eta(I^1), \eta(I^2)} \circ \widehat{f}_{\eta(I^0), \eta(I^2), \eta(I^3)} \circ \dots \circ \widehat{f}_{\eta(I^0), \eta(I^i), \eta(I^{i+1})} \circ \dots \circ \widehat{f}_{\eta(I^0), \eta(I^{k-1}), \eta(I^k)}.$$

This juxtaposition gives rise to an  $(k+1)$ -gon  $\varphi \in \pi_2(\widehat{\Theta}_1, \dots, \widehat{\Theta}_k, \Theta)$ , with  $\mathcal{D}(\varphi) \geq 0$  and  $n_z(\varphi) = 0$ , where here  $\Theta \in \mathbb{T}_{\eta(I^0)} \cap \mathbb{T}_{\eta(I^k)}$ . By additivity of the Maslov index, this  $k+1$ -gon has  $\mu(\varphi) = k-2$ ; it is also easily seen to have  $n_z(\phi) = 0$ . It follows that there are no  $k+1$ -gons  $\varphi' \in \pi_2(\widehat{\Theta}_1, \dots, \widehat{\Theta}_k, \widehat{\Theta}')$  with  $\mu(\varphi') = 0$  and  $\mathcal{D}(\phi') \geq 0$ .

Consider now the case where  $k = 2$ . In this case,  $I$  and  $J$  differ in at least one place, and at most two.

If  $I$  and  $J$  differ in one place, a direct inspection of the Heegaard triple (which leads to the “blowup formula” in [15]) shows that the maps cancel in pairs. This is spelled out in Proposition 9.5 of [14].

If they differ in two places, there are two choices for  $I^1$  with  $I = I^0 < I^1 < J = I^2$ . For each possible  $I^1$ , it is the case that

$$\widehat{f}_{\eta(I^0), \eta(I^1), \eta(I^2)}(\widehat{\Theta}_1 \otimes \widehat{\Theta}_2) = \pm \widehat{\Theta}_3.$$

One can see this by explicitly drawing the Heegaard triple, which splits into torus summands, as in [15]. (See also Proposition 6.1 below.) Since there are two choices, they cancel in the sum, modulo two.

Indeed, it is straightforward to give all triangles orientations so that the cancellation takes place with coefficients in  $\mathbb{Z}$ , as well. For example, for fixed  $I$  and  $J$  which differ in two places, and for which we have a sequence of immediate successors of the form  $I < K < J$ ; we have exactly one other sequence of immediate successors  $I < L < J$ . Label these two possible intermediate mult-framings so that  $K < L$ . We then define orientations on the triangles (and choose our canonical generators for all the  $Y_{\eta(I), \eta(I')}$ ) so that

$$\widehat{f}_{\eta(I), \eta(K), \eta(J)}(\widehat{\Theta}_1 \otimes \widehat{\Theta}_2) = \widehat{\Theta}_3,$$

while

$$\widehat{f}_{\eta(I), \eta(L), \eta(J)}(\widehat{\Theta}_1 \otimes \widehat{\Theta}_2) = -\widehat{\Theta}_3.$$

In the case where  $k = 1$ , the stated relation is simply the one that  $\widehat{\Theta}_1$  is a cycle.  $\square$

**Proposition 4.6.** *The map  $D$  from Equation (10) satisfies  $D^2 = 0$ .*

**Proof.** This follows from the associativity formula (Equation (9)) for the Heegaard tuple  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I^1), \dots, \boldsymbol{\eta}(I^k), z)$ , together with Lemma 4.5. Specifically, according to that lemma, the only degenerations in Equation (9) which do not contribute 0 to the sum are the ones which involve  $\boldsymbol{\alpha}$  in both polygons. Those, in turn, are the various components of  $D^2$ .

Strictly speaking, for this result to hold over  $\mathbb{Z}$  (not just over  $\mathbb{Z}/2\mathbb{Z}$ ), note that Lemma 4.5 holds for particular orientations on the spaces of triangles for

$$(\Sigma, \eta(I^0), \eta(I^1), \eta(I^2)),$$

where here  $I^0 < I^1 < I^2$  are immediate successors. In fact, it is straightforward to see that such a system of orientations can be used to induce a coherent orientation on all the polygons, so that associativity still holds, and hence the result holds over  $\mathbb{Z}$ , as well.  $\square$

In view of Proposition 4.6, we can think of  $X$  as a chain complex, endowed with the differential  $D$ . We can define some other associated complexes as follows. If  $S \subset \{0, 1, \infty\}^\ell$  is a subset with the property that for each  $I, J \in S$ , for all  $K \in \{0, 1, \infty\}^\ell$  with  $I < K < J$ , we also have that  $K \in S$ , then we let  $X(S)$  denote the group  $\bigoplus_{I \in S} \widehat{CF}(Y(I))$  endowed with the differential naturally induced by  $D$ .

With all the notational background, we are now ready to prove a strong form of the surgery long exact sequence for a single knot in a three-manifold  $Y$ .

**Theorem 4.7.** *Let  $K$  be a framed knot in a three-manifold  $Y$ , and let*

$$\widehat{f}: \widehat{CF}(Y_0(K)) \longrightarrow \widehat{CF}(Y_1(K))$$

*denote the chain map induced by the cobordism. Then, the chain complex  $\widehat{CF}(Y)$  is quasi-isomorphic to the mapping cone of  $\widehat{f}$ .*

**Proof.** To start, let  $(\Sigma, \alpha, \beta, \gamma, \delta, z)$  denote the associated Heegaard quintuple. In particular,  $Y_{\alpha, \beta}$ ,  $Y_{\alpha, \gamma}$ ,  $Y_{\alpha, \delta}$  describe  $Y$ ,  $Y_0$ , and  $Y_1$  respectively, and the remaining three-manifolds on the boundary describe  $\#^{g-1}(S^2 \times S^1)$ . Indeed, to fit precisely with the hypotheses of that lemma, we choose infinitely many copies of the  $g$ -tuples  $\beta$ ,  $\gamma$ , and  $\delta$  (denoted  $\beta^{(i)}$ ,  $\gamma^{(i)}$ ,  $\delta^{(i)}$  for  $i \in \mathbb{Z}$ ), all of which are generic exact Hamiltonian perturbations of one another, in the interest of admissibility (in the sense of Subsection 4.2.2 of [17]).

In this case, the chain map we described earlier  $X$  splits (as a module) as  $\widehat{CF}(Y_0) \oplus \widehat{CF}(Y_1) \oplus \widehat{CF}(Y)$ , and its differential decomposes as

$$(11) \quad \partial = \begin{pmatrix} D_0 & 0 & 0 \\ D_{0<1} & D_1 & 0 \\ D_{0<1<\infty} & D_{1<\infty} & D_\infty \end{pmatrix}.$$

Letting  $\widehat{CF}(Y)$ ,  $\widehat{CF}(Y_0)$ , and  $\widehat{CF}(Y_1)$  play the roles of  $A_1$ ,  $A_2$ , and  $A_3$  respectively, the various components of the differential play the roles of the  $f_i$  and  $H_i$  (compare Equations (11) and (8)),

Indeed,  $A_{3i+1}$ ,  $A_{3i+2}$  and  $A_{3i+3}$  all represent  $\widehat{CF}(Y_0)$ ,  $\widehat{CF}(Y_1)$  and  $\widehat{CF}(Y)$  respectively, only now we use the various translates of the  $\beta$ ,  $\gamma$ , and  $\delta$ ; in particular  $A_{3i+1}$  is the Floer complex  $\widehat{CF}(\alpha, \beta^{(i)})$ .

Hypothesis (1) of Lemma 4.4 follows at once from the fact that  $D$  is a chain complex (Proposition 4.6).

It remains to verify Hypothesis (2) of Lemma 4.4.



Let  $\theta_i$  be the chain homotopy equivalences induced by equivalences of Heegaard diagrams; e.g.  $\theta_{3i}$  is the chain map  $\widehat{CF}(\alpha, \beta^{(i)}) \longrightarrow \widehat{CF}(\alpha, \beta^{(i+1)})$  obtained by product with the canonical generator  $\widehat{\Theta}_{\beta^{(i)}, \beta^{(i+1)}}$ .

We claim that

$$f_3 \circ H_1 + H_2 \circ f_1: A_1 \longrightarrow A_4$$

is chain homotopic to  $\theta_1$ , and the chain homotopy is given by

$$\mathbf{x} \mapsto f_{\alpha, \beta, \gamma, \delta, \beta^{(i)}}(\mathbf{x} \otimes \widehat{\Theta}_{\beta, \gamma} \otimes \widehat{\Theta}_{\gamma, \delta} \otimes \widehat{\Theta}_{\delta, \beta^{(1)}}).$$

This in turn follows at once from associativity, together with the fact that

$$(12) \quad f_{\beta, \gamma, \delta, \beta^{(1)}}(\widehat{\Theta}_{\beta, \gamma} \otimes \widehat{\Theta}_{\gamma, \delta} \otimes \widehat{\Theta}_{\delta, \beta^{(1)}}) = \widehat{\Theta}_{\beta, \beta^{(1)}}.$$

This latter equality follows from a direct inspection of the Heegaard diagram for the quadruple  $(\Sigma, \beta, \gamma, \delta, \beta^{(1)}, z)$ . (i.e. the count of pseudo-holomorphic pentagons), as illustrated in Figures 8 and 9.

In Figure 8, we consider the special case where the genus  $g = 1$ . In the picture, and in the following discussion,  $\beta_1^{(1)}$  is denoted  $\beta'_1$ . The four corners of the shaded quadrilateral are the canonical generators  $\widehat{\Theta}_{\beta_1, \gamma_1}$ ,  $\widehat{\Theta}_{\gamma_1, \delta_1}$ ,  $\widehat{\Theta}_{\delta_1, \beta'_1}$ , and  $\widehat{\Theta}_{\beta'_1, \beta_1}$  (read in clockwise order). Indeed, it is straightforward to see (by passing to the universal cover), that the shaded quadrilateral represents the only homotopy class  $\varphi_1$  of Whitney quadrilaterals

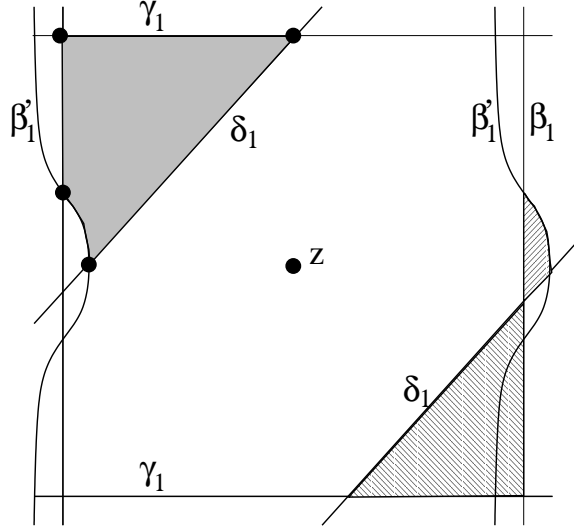


FIGURE 8. **A holomorphic quadrilateral.** The shaded quadrilateral has a unique holomorphic representative (by the Riemann mapping theorem), while the one indicated with the hatching does not, as it has both positive and negative local multiplicities, as indicated by the two directions in the hatching.

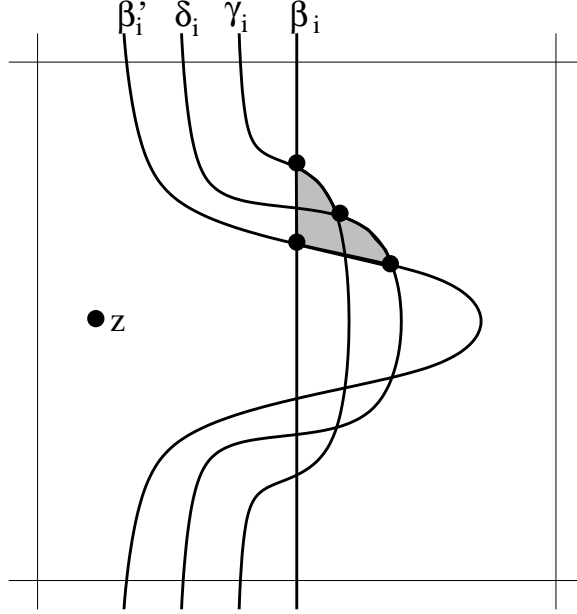


FIGURE 9. **Other factors of the holomorphic quadrilateral.** We have illustrated here a Heegaard quadruple (in a genus one surface) whose four boundary components are  $S^2 \times S^1$ . In the homotopy class indicated by the shaded quadrilateral  $\varphi_i \in \pi_2(\hat{\Theta}_{\beta_i, \gamma_i}, \hat{\Theta}_{\gamma_i, \delta_i}, \hat{\Theta}_{\delta_i, \beta'_i}, \hat{\Theta}_{\beta'_i, \beta_i})$ , there is a moduli space of pseudo-holomorphic quadrilaterals which is clearly one-dimensional, parameterized by a cut at the vertex where  $\gamma_i$  and  $\delta_i$  meet. We take the connected sum of  $g - 1$  copies of this picture (at the reference point  $z$ ) with the picture illustrated in Figure 8 to obtain the general case of the quadrilateral considered in the proof of Theorem 4.7.

with  $n_z(\varphi_1) = 0$  and all of whose local multiplicities are non-negative. By the Riemann mapping theorem, now, this homotopy class  $\varphi_1$  has a unique holomorphic representative  $u_1$ . (By contrast, we have also pictured here another Whitney quadrilateral with hatchings, whose local multiplicities are all 0, +1, and -1; +1 at the region where the hatchings go in one direction and -1 where they go in the other.)

For the general case ( $g > 1$ ), we take the connected sum of the case illustrated in Figure 8 with  $g - 1$  copies of the torus illustrated in Figure 9. In this picture, we have illustrated the four curves  $\beta_i, \gamma_i, \delta_i, \beta'_i$  for  $i > 1$ , which are Hamiltonian translates of one another. Now, there is a homotopy class of quadrilateral  $\varphi_i \in \pi_2(\hat{\Theta}_{\beta_i, \gamma_i}, \hat{\Theta}_{\gamma_i, \delta_i}, \hat{\Theta}_{\delta_i, \beta'_i}, \hat{\Theta}_{\beta'_i, \beta_i})$ , and a forgetful map  $\mathcal{M}(\phi) \rightarrow \mathcal{M}(\square)$  which remembers only the conformal class of the domain (where here  $\mathcal{M}(\square)$  denotes the module space of conformal classes of disks with four marked boundary points, also referred to simply as quadrilaterals). Both moduli spaces are one-dimensional (the first moduli space is

parameterized by the length of the cut into the region, while the second is parameterized by the ratio of the length to the width, after the quadrilateral is uniformized to a rectangle). By Gromov's compactness theorem, the forgetful map is proper; and it is easy to see that this it has degree one, and hence for some generic conformal class of quadrilateral, there is a unique pseudo-holomorphic quadrilateral whose domain has the specified conformal class. Now, letting  $u_1$  (the pseudo-holomorphic representative of the homotopy class  $\varphi_1$  described in the previous paragraph) determine the conformal class of the rectangle, we let  $u_i$  for  $i > 1$  be the pseudo-holomorphic representatives for  $\phi_i$  whose domain supports the same conformal class. Then  $u_1 \times \dots \times u_g \in \varphi_1 \times \dots \times \varphi_g$  is easily seen to be the unique holomorphic quadrilateral in  $\pi_2(\hat{\Theta}_{\beta,\gamma}, \hat{\Theta}_{\gamma,\delta}, \hat{\Theta}_{\delta,\beta'}, \hat{\Theta}_{\beta',\beta})$ , hence proving Equation (12) which, in turn, yields Hypothesis (2) of Lemma 4.4. The theorem now follows directly from Lemma 4.4.  $\square$

We now turn to Theorem 4.1.

**Proof of Theorem 4.1.** The theorem is established by induction on the number of components of the link. The case where the link has a single component is a direct consequence of Theorem 4.7.

We form the chain complex  $X$  as before. We claim first that  $H_*(X) = 0$ .

Let  $S$  be any subset of  $\{0, 1, \infty\}^\ell$  with the following property. For each  $\sigma \in S$ , there are two other elements  $\tau, \nu \in S$ , with the property that the three subsets  $\sigma, \tau, \nu$  are three distinct elements, and they all differ in a single component. It is a straightforward application of Theorem 4.7 (or, more precisely, the reformulation of the mapping cone lemma, Lemma 4.4, described after its proof) that  $H_*(X(S)) = 0$ . In particular,  $H_*(X) = 0$ .

In particular, we can let

$$S = \{0, 1\}^{\ell-1} \times \{0, 1, \infty\}.$$

By the above criterion,  $H_*(X(S)) = 0$ . Indeed, the letting  $T = \{0, 1\}^\ell$ , we have a short exact sequence

$$0 \longrightarrow X(T) \xrightarrow{f} X(S) \xrightarrow{g} X(\{0, 1\}^{\ell-1} \times \{\infty\}) \longrightarrow 0$$

from which it follows at once that the connecting homomorphism induces an isomorphism in homology

$$H_*(X(\{0, 1\}^{\ell-1} \times \{\infty\})) \xrightarrow{\cong} H_*(X(\{0, 1\}^\ell)).$$

By our inductive hypothesis, it follows that  $H_*(X(\{0, 1\}^{\ell-1} \times \{\infty\})) \cong \widehat{HF}(Y)$ , completing the proof.  $\square$

## 5. KHOVANOV'S INVARIANTS

We briefly describe here Khovanov's categorification of the Jones polynomial; for more details, see [11] and [2]. We make some simplifying assumptions here: we will use coefficients in  $\mathbb{Z}/2\mathbb{Z}$  throughout, and we specialize to the case where  $U = 0$  (in Khovanov's sense). Our notation and exposition are tailored to fit neatly into the context of the present paper. In particular, the groups we describe here are actually the Khovanov homology of the mirror of  $L$ .

Let  $X = S_1 \cup \dots \cup S_k$  be a collection of disjoint embedded, simple closed curves in the plane. Let  $Z(X)$  denote the  $\mathbb{Z}/2\mathbb{Z}$ -vector space, formally generated by the components  $[S_1], \dots, [S_k]$ , and let  $V(X)$  denote exterior algebra

$$V(X) = \Lambda^* Z(X),$$

i.e. this is the quotient of the polynomial algebra over  $\mathbb{Z}/2\mathbb{Z}$  generated by  $[S_i]$ , divided out by the relations  $[S_i]^2 = 0$  for  $i = 1, \dots, k$ . (When comparing the notation used in the discussion here with that of [11], observe that the element  $[S_1] \wedge \dots \wedge [S_{m_\ell}] \in V(X)$ , where  $\{m_j\}_{j=1}^\ell$  is a subsequence of  $\{1, \dots, k\}$  corresponds to the element

$$v_1^{\epsilon_1} \otimes \dots \otimes v_k^{\epsilon_k}$$

where here  $\epsilon_i \in \{\pm\}$  is obtained by

$$\epsilon_i = \begin{cases} - & \text{if } i \in \{m_j\}_{j=1}^\ell \\ + & \text{otherwise} \end{cases}$$

in Khovanov's notation, c.f. [11].)

Next, consider a pair of pants, thought of as a morphism from  $X = S_1 \cup \dots \cup S_k \cup S_{k+1}$  to a new submanifold  $X' = S_1 \cup \dots \cup S_{k-1} \cup S'_k$  containing a component  $S'_k$  which is obtained by merging  $S_k$  and  $S_{k+1}$ . In this case, we have a natural identification

$$Z(X') = Z(X)/[S_k] \sim [S_{k+1}],$$

and correspondingly natural isomorphisms

$$\alpha: (S_{k+1} - S_k) \wedge V(X) \xrightarrow{\cong} V(X') \quad \text{and} \quad \beta: V(X') \xrightarrow{\cong} V(X)/(S_{k+1} - S_k) \wedge V(X).$$

We then define the multiplication

$$m: V(X) \longrightarrow V(X')$$

to be the composite

$$V(X) \xrightarrow{(S_{k+1}-S_k) \wedge \cdot} (S_{k+1} - S_k) \wedge V(X) \xrightarrow{\alpha} V(X').$$

By reversing the orientation of the “pair of pants”, we have a morphism from  $X'$  to  $X$ , instead. In this case, we have a comultiplication

$$\Delta: V(X') \longrightarrow V(X)$$

induced by the composition

$$V(X') \xrightarrow{\beta} \frac{V(X)}{(S_{k+1}-S_k) \wedge V(X)} \xrightarrow{(S_{k+1}-S_k) \wedge} V(X).$$

Let  $L$  be a link, and fix a generic projection of  $L$ ,  $\mathcal{D}$ , with  $\ell$  double points. One can form resolutions indexed by subsets  $I \in \{0, 1\}^\ell$ . Specifically, for each  $I$ ,  $\mathcal{D}(I)$  is a disjoint union of circles in the plane. If  $I'$  is an immediate successor of  $I$ , then  $\mathcal{D}(I')$  differs from  $\mathcal{D}(I)$  by a single pair of pants. We have a map

$$\mathfrak{d}_{I < I'}: V(\mathcal{D}(I)) \longrightarrow V(\mathcal{D}(I')),$$

given by multiplication or co-multiplication, according to whether  $\mathcal{D}(I')$  has one fewer or one more component than  $\mathcal{D}(I)$ .

Fix a diagram  $\mathcal{D}$  for an oriented link  $L$ , and let  $n_+(\mathcal{D})$  resp.  $n_-(\mathcal{D})$  denote the number of positive resp. negative crossings for the link  $L$ , according to the usual conventions (c.f. Figure 10). Consider next the graded Abelian group

$$\text{CKh}(\mathcal{D}, m) = \bigoplus_{\{I \in \{0,1\}^\ell \mid \sum_{i=1}^\ell I_i + n_+(\mathcal{D}) = m\}} V(\mathcal{D}(I)),$$

endowed with the differential

$$\mathfrak{d}: \text{CKh}(\mathcal{D}, m) \longrightarrow \text{CKh}(\mathcal{D}, m-1)$$

whose restriction to  $V(\mathcal{D}(I)) \subset \text{CKh}(\mathcal{D}, m)$  is the sum

$$\mathfrak{d} = \sum_{I < I'} \mathfrak{d}_{I < I'},$$

where the sum is taken over all immediate successors  $I'$  of  $I$ . In each dimension  $m$ ,  $\text{CKh}(\mathcal{D}, m)$  is endowed with an additional grading, the “q-grading”, defined by the splitting

$$V(\mathcal{D}(I)) = \bigoplus_{n \in \mathbb{Z}} V_n(\mathcal{D}(I)),$$

where

$$V_n(\mathcal{D}(I)) = \Lambda^k Z^*(\mathcal{D}(I)),$$

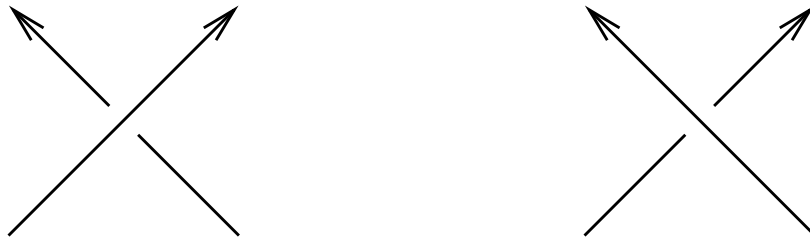


FIGURE 10. **Crossing conventions.** Crossings of the first kind are positive, and those of the second kind are negative.

where

$$n = \dim Z(\mathcal{D}) - 2k - n_-(\mathcal{D}) + 2n_+(\mathcal{D}).$$

Correspondingly, we write

$$\mathrm{CKh}(\mathcal{D}) = \bigoplus_{m,n \in \mathbb{Z}} \mathrm{CKh}(\mathcal{D}, m, n).$$

Note that  $\mathfrak{d}$  carries  $\mathrm{CKh}(\mathcal{D}, m, n)$  to  $\mathrm{CKh}(\mathcal{D}, m-1, n-1)$ .

It is easy to see that  $\mathfrak{d}^2 = 0$ . Khovanov's homology of the mirror of  $L$  is the Abelian group

$$\mathrm{Kh}(r(L)) = H_*(\mathrm{CKh}_*(\mathcal{D}), \mathfrak{d}),$$

thought of as bi-graded Abelian group

$$\mathrm{Kh}(r(L)) = \bigoplus_{m,n \in \mathbb{Z} \oplus \mathbb{Z}} \mathrm{Kh}(\mathcal{D}, m, n).$$

Note that the complex  $\mathrm{CKh}(\mathcal{D})$  depends on the projection of  $L$ . Khovanov shows, however, that the homology of this complex is independent of this choice, i.e.  $\mathrm{Kh}(L)$  is a link invariant. Moreover, he shows that these groups satisfy a skein exact sequence

$$\dots \longrightarrow \mathrm{Kh}(r(L)) \longrightarrow \mathrm{Kh}(r(L_0)) \longrightarrow \mathrm{Kh}(r(L_1)) \longrightarrow \dots$$

Khovanov's theory is related to the Jones polynomial by the formula

$$\widehat{J}(r(L)) = \sum_{m,n} (-1)^m \left( \mathrm{rk} \, \mathrm{Kh}(L, m, n) \right) \cdot q^n,$$

where here  $\widehat{J}(L) \in \mathbb{Z}[q, q^{-1}]$  is the unnormalized Jones polynomial of the link  $K$ , characterized by the formulas

$$\begin{aligned} \widehat{J}(\emptyset) &= 1 \\ \widehat{J}((\text{unknot}) \cup L) &\doteq (q + q^{-1}) \cdot \widehat{J}(L) \\ \widehat{J}(r(L)) &\doteq \widehat{J}(r(L_0)) - q \cdot \widehat{J}(r(L_1)), \end{aligned}$$

where in the last equation,  $L_0$  and  $L_1$  are taken with respect to the two resolutions at any double-point of any projection of  $L$ , and where for  $f, g \in \mathbb{Z}[q, q^{-1}]$ , we write  $f \doteq g$  if  $f = q^j \cdot g$  for some  $j \in \mathbb{Z}$ .

In [12], Khovanov gives a modification of the above constructions to define a “reduced” theory  $\widetilde{\mathrm{CKh}}(L)$ , which is related to the normalized Jones polynomial  $J(L)$  is defined by  $(q + q^{-1}) \cdot J(L) = \widehat{J}(L)$ . For the reduced theory, one marks a generic point in the projection of  $L$ , so that now in all the various resolutions, there is always a distinguished circle. The reduced Khovanov complex is the quotient of  $\widetilde{\mathrm{CKh}}(\mathcal{D})$  by the subcomplex of  $\mathrm{CKh}(\mathcal{D})$  given by

$$\bigoplus_{I \in \{0,1\}^\ell} [S_I] \wedge V(\mathcal{D}(I))$$

where here  $S_I$  is the component in  $\mathcal{D}(I)$  which contains the marked point. This gives a chain complex whose which splits into summands indexed by  $I \in \{0, 1\}^\ell$ , and the corresponding summand is denoted

$$\tilde{V}(\mathcal{D}(I)) = V(\mathcal{D}(I))/[S_I] \wedge V(\mathcal{D}(I)).$$

For this construction.

$$J(r(L)) = \sum_{m,n} (-1)^m \left( \text{rk } \widetilde{\text{Kh}}(L, m, n) \right) \cdot q^n,$$

c.f. [12].

We have described Khovanov's construction with  $\mathbb{Z}/2\mathbb{Z}$  coefficients. In fact, Khovanov's original definition from [11] make sense with coefficients in  $\mathbb{Z}$ . In this case, however the reduced homology (described in [12]) depends on the link, together with the distinguished component containing the marked point. If one takes coefficients in  $\mathbb{Z}/2\mathbb{Z}$ , as we have here, it is easy to see that the reduced theory is independent of this additional choice, hence giving a link invariant.

## 6. THE SPECTRAL SEQUENCE FOR A BRANCHED DOUBLE COVER

Throughout this section, we fix our coefficient ring to be  $\mathbb{Z}/2\mathbb{Z}$ : i.e. if  $Y$  is a three-manifold,  $\widehat{HF}(Y)$  will denote  $\widehat{HF}$  of  $Y$  with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ , and similarly,  $H_*(Y)$  will denote singular homology with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ .

In comparing Khovanov's homology with  $\widehat{HF}$ , we rely on the following (fairly straightforward) result about  $\widehat{HF}$ , proved in earlier papers. For the statement, note that if  $Y$  is any three-manifold, then  $\widehat{HF}(Y)$  is a module over the algebra  $\Lambda^*H_1(Y)/\text{Tors}$ .

**Proposition 6.1.** *Let  $Y \cong \#^k(S^2 \times S^1)$ . Then,  $\widehat{HF}(Y)$  is a rank one, free module over the ring  $\Lambda^*H_1(Y)$ , generated by some class  $\Theta \in \widehat{HF}(Y)$ . Moreover, if  $K \subset Y$  is a curve which is dual to one of the circles in one of the  $S^2 \times S^1$  summands, then the three-manifold  $Y' = Y_0(K)$  is diffeomorphic to  $\#^{k-1}(S^2 \times S^1)$ , with a natural identification*

$$\pi: H_1(Y)/[K] \longrightarrow H_1(Y').$$

Under the cobordism  $W$  induced by the two-handle, the map

$$F_W: \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y')$$

is specified by

$$F_W(\xi \cdot \Theta) = \pi(\xi) \cdot \Theta',$$

where here  $\Theta'$  is some fixed generator of  $\widehat{HF}(Y')$ , and  $\xi$  is any element of  $\Lambda^*H_1(Y)$ . Dually, if  $K \subset Y$  is an unlink, then  $Y'' = Y_0(K) \cong \#^{k+1}(S^2 \times S^1)$ , with a natural inclusion

$$i: H_1(Y) \longrightarrow H_1(Y'').$$

Under the cobordism  $W'$  induced by the two-handle the map

$$F_{W'}: \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y'')$$

is specified by

$$F_{W'}(\xi \cdot \Theta) = \xi \wedge [K''] \cdot \Theta'',$$

where here  $[K''] \in H_1(Y''; \mathbb{Z})$  is a generator in the kernel of the map  $H_1(Y'') \longrightarrow H_1(W')$ .

**Proof.** The identification of  $\widehat{HF}(Y)$  follows from a direct inspection of the Heegaard diagram, as explained in Subsection 3.1. The fact that  $F_W(\Theta)$  is a generator for  $\widehat{HF}(Y')$  (which we denote by  $\Theta'$ ) follows from a direct inspection of a Heegaard triple which naturally splits into genus one summands, c.f. [15]. (Alternately, one could use the surgery exact sequence which in this case reads

$$\dots \longrightarrow \widehat{HF}(Y') \longrightarrow \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y') \longrightarrow \dots$$

to deduce that the map from  $\widehat{HF}(Y)$  to  $\widehat{HF}(Y')$ , which in this case is induced by the two-handle  $W$  equipped with its torsion  $\text{Spin}^c$  structure, is surjective.) The more



general formula for  $F_W$  follows from naturality of the triangle maps under the  $H_1$  action (c.f. [15]). The case of  $W'$  follows similarly.  $\square$

See [17] and [15].

We can now link Khovanov's construction (using notation from Section 5) with  $\widehat{HF}$ :

**Proposition 6.2.** *Fix a projection  $\mathcal{D}$  for  $K$ . There is an isomorphism for each  $I$*

$$\Psi(I): \tilde{V}(\mathcal{D}(I)) \xrightarrow{\cong} \widehat{HF}(\Sigma(\mathcal{D}(I))),$$

*which is natural under cobordisms, in the following sense. If  $I'$  is an immediate successor of  $I$ , then there is a naturally induced cobordism (induced from a single two-handle addition) from  $\Sigma(\mathcal{D}(I))$  to  $\Sigma(\mathcal{D}(I'))$ , and hence an induced map*

$$\widehat{G}_{I < I'}: \widehat{HF}(\Sigma(I)) \longrightarrow \widehat{HF}(\Sigma(I')).$$

*Naturality of  $\Psi$  is captured in the following commutative diagram, which is valid whenever  $I'$  is an immediate successor of  $I$ :*

$$(13) \quad \begin{array}{ccc} \tilde{V}(\mathcal{D}(I)) & \xrightarrow{\mathfrak{d}_{I < I'}} & \tilde{V}(\mathcal{D}(I')) \\ \psi(I) \downarrow & & \downarrow \psi(I') \\ \widehat{HF}(\Sigma(\mathcal{D}(I))) & \xrightarrow{G_{I < I'}} & \widehat{HF}(\Sigma(\mathcal{D}(I'))) \end{array}$$

**Proof.** First, note that for each  $I$ , we can write  $\mathcal{D}(I) = S_0 \cup \dots \cup S_k$ , where here the  $S_i$  are pairwise disjoint unknots, and  $k \geq 0$ . In this case,  $\Sigma(\mathcal{D}(I)) \cong \#^k(S^2 \times S^1)$ . Indeed, we give a basis  $\{[\gamma_i]\}_{i=1}^k$  for  $H_1(\Sigma(\mathcal{D}(I)))$  as follows. For  $i > 0$ , let  $[\gamma_i] \in H_1(\Sigma(\mathcal{D}(I)))$  be the homology class of the curve obtained as the branched double cover of an arc from  $S_0$  to  $S_i$  (recall that we are using here  $\mathbb{Z}/2\mathbb{Z}$  coefficients). This induces the identification

$$\tilde{Z}(\mathcal{D}(I)) \cong H_1(\Sigma(\mathcal{D}(I))).$$

Combined with Proposition 6.1, we get a canonical identification

$$\tilde{V}(\mathcal{D}(I)) \cong \widehat{HF}(\Sigma(\mathcal{D}(I))).$$

Commutativity of Diagram (13) is proved in four cases, each of which follows from Proposition 6.1.

Suppose that  $I'$  is obtained from  $I$  by merging two circles  $S_1$  and  $S_2$ , neither of which is marked. Then, we claim that in the cobordism  $W$ , the curves  $\gamma_1$  and  $\gamma_2$  become homologous; indeed, both are homologous to the new curve  $\gamma'_1$ . Commutativity of the square now follows readily from Proposition 6.1 and the definition of  $\mathfrak{d}_{I < I'}$ .

Dually, when  $I'$  is obtained from  $I$  by splitting an unmarked circle  $T_1$  into two circles  $S_1$  and  $S_2$ , the curve  $\gamma_1 - \gamma_2$  is null-homologous in the induced cobordism  $W'$ . Again, commutativity of the claimed square now follows readily from Proposition 6.1.

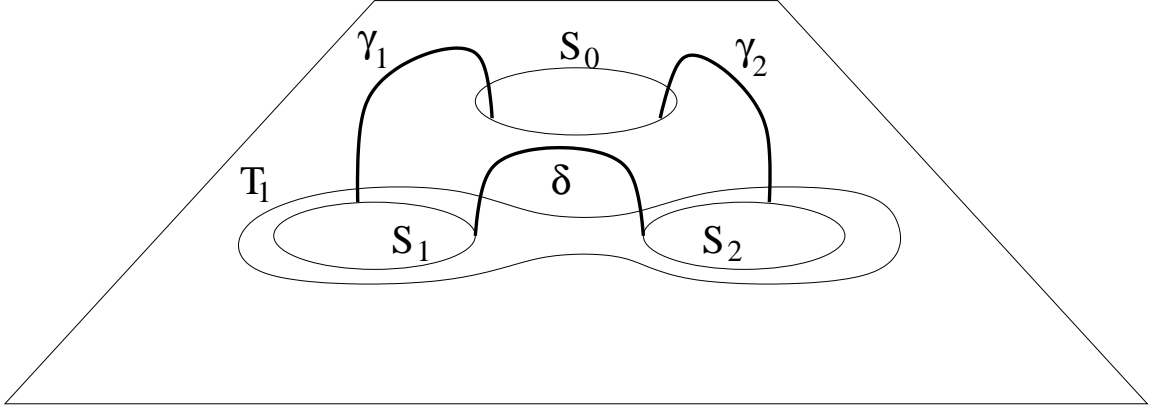


FIGURE 11. **Homological relations in the cobordisms.** The plane of the (un)-link projection is indicated by the quadrilateral, on which we have the marked component  $S_0$ , two unlink components  $S_1$  and  $S_2$ , and an alternative component  $T_1$ , obtained by merging  $S_1$  and  $S_2$ . We have also illustrated the curves  $\gamma_1$  and  $\gamma_2$ . The picture illustrates that  $\gamma_1$  and  $\gamma_2$  become homologous after  $S_1$  and  $S_2$  are merged. Dually, it illustrates that in the cobordism where  $T_1$  is divided in two, it is the curve  $\delta = \gamma_1 - \gamma_2$  which becomes null-homologous.

The two corresponding cases involving a marked circle follow similarly (indeed, they follow formally in the same manner, once we declare  $[\gamma_0] = 0$ ).  $\square$

With these preliminaries in place, we can now state and prove the following precise version of Theorem 1.1 in the introduction.

**Theorem 6.3.** *Given a projection  $\mathcal{D}$  of a link  $L$ , there is a spectral sequence converging to  $\widehat{HF}(\Sigma(L))$  whose  $(E^1, d^1)$  complex is isomorphic to Khovanov's reduced chain complex (for the mirror of  $L$ ); i.e. there are isomorphisms  $\psi^m$  making the following diagram commute:*

$$\begin{array}{ccc}
 E_{m-n_+(\mathcal{D})}^1 & \xrightarrow{d_{m-n_+(\mathcal{D})}^1} & E_{m-n_+(\mathcal{D})-1}^1 \\
 \psi^m \downarrow & & \psi^{m-1} \downarrow \\
 \widetilde{\text{CKh}}(\mathcal{D}(L), m) & \xrightarrow{\partial^m} & \widetilde{\text{CKh}}(\mathcal{D}(L), m-1).
 \end{array}$$

In particular, the  $E^2$  term of this sequence is identified with Khovanov's homology of  $L$ .

**Proof.** As explained in Section 2, a diagram  $\mathcal{D}$  for a link  $K$  with  $\ell$  crossings gives rise to a link  $L$  in  $Y = \Sigma(K)$  whose components correspond to the crossings of  $\mathcal{D}$ . Moreover, for each  $I \subset \{0, 1\}^\ell$ , the three-manifold obtained by performing surgeries along these

components of  $L$  is the branched double cover of  $S^3$  branched along the collection of unknots  $\mathcal{D}(I)$ . We now apply Theorem 4.1. The identification with Khovanov's (reduced) complex is now provided by Proposition 6.2. Note that the spectral sequence coming from Theorem 4.1 is filtered by a cube, rather than  $\mathbb{Z}$ . We pass to a  $\mathbb{Z}$ -filtered object by “flattening” the cube as usual (c.f. Remark 4.2).  $\square$

**Proof of Corollary 1.2.** The inequalities follow from  $\det(L) = |H^2(\Sigma(L); \mathbb{Z})|$  (c.f. [13]); but this agrees with  $\chi(\widehat{HF}(\Sigma(L)))$  (c.f. Proposition 5.1 of [14]). It follows at once that  $\det(L) \leq \text{rk} \widehat{HF}(\Sigma(L))$ . The other inequality follows from Theorem 6.3, together with the straightforward inequality for spectral sequences  $\text{rk} E^\infty \leq \text{rk} E^2$ .  $\square$

## REFERENCES

- [1] C. C. Adams, J. F. Brock, J. Bugbee, and et al. Almost alternating links. *Topology Appl.*, 46(2):151–165, 1992.
- [2] D. Bar-Natan. On Khovanov’s categorification of the Jones polynomial. *Algebraic and Geometric Topology*, 2:337–370, 2002.
- [3] R. Crowell. Genus of alternating link types. *Ann. of Math. (2)*, 69:258–275, 1959.
- [4] V. de Silva. *Products in the symplectic Floer homology of Lagrangian intersections*. PhD thesis, Oxford University, 1999.
- [5] A. Floer. A relative Morse index for the symplectic action. *Comm. Pure Appl. Math.*, 41(4):393–407, 1988.
- [6] A. Floer. Instanton homology and Dehn surgery. In H. Hofer, C. H. Taubes, A. Weinstein, and E. Zehnder, editors, *The Floer Memorial Volume*, number 133 in Progress in Mathematics, pages 77–97. Birkhäuser, 1995.
- [7] K. A. Frøyshov. The Seiberg-Witten equations and four-manifolds with boundary. *Math. Res. Lett.*, 3:373–390, 1996.
- [8] K. Fukaya, Y-G. Oh, K. Ono, and H. Ohta. *Lagrangian intersection Floer theory—anomaly and obstruction*. Kyoto University, 2000.
- [9] V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985.
- [10] L. H. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.
- [11] M. Khovanov. A categorification of the Jones polynomial. math.QA/9908171, 1999.
- [12] M. Khovanov. Patterns in knot cohomology I. math.QA/0201306, 2002.
- [13] W. B. Raymond Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, 1997.
- [14] P. S. Ozsváth and Z. Szabó. Holomorphic disks and three-manifold invariants: properties and applications. math.SG/0105202, To appear in *Annals of Math.*
- [15] P. S. Ozsváth and Z. Szabó. Holomorphic triangles and invariants for smooth four-manifolds. math.SG/0110169.
- [16] P. S. Ozsváth and Z. Szabó. On knot Floer homology and lens space surgeries. math.GT/0303017.
- [17] P. S. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. math.SG/0101206. To appear in *Annals of Math.*, 2001.
- [18] P. S. Ozsváth and Z. Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Advances in Mathematics*, 173(2):179–261, 2003.
- [19] P. S. Ozsváth and Z. Szabó. On the Floer homology of plumbed three-manifolds. *Geometry and Topology*, 7:185–224, 2003.
- [20] D. Salamon and E. Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.*, 45(10):1303–1360, 1992.
- [21] G. Tian. Quantum cohomology and its associativity. In R. Bott, M. Hopkins, A. Jaffe, I. Singer, D. Stroock, and S-T. Yau, editors, *Current Developments in Mathematics, 1995*, pages 361–401. Internat. Press, 1994.
- [22] S. Wolfram. *The Mathematica book. Fourth edition*. Wolfram Media, Inc., 1999.

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